Economics & Finance for Systems Design

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L1. Introduction

What is Economics?

Economics is the study of how limited resources can be used to satisfy (theoretically unlimited) human needs.

Individuals and societies choose to use scarce resources that nature and previous generations have provided.

What is the Engineering Economy?

The engineering economy:

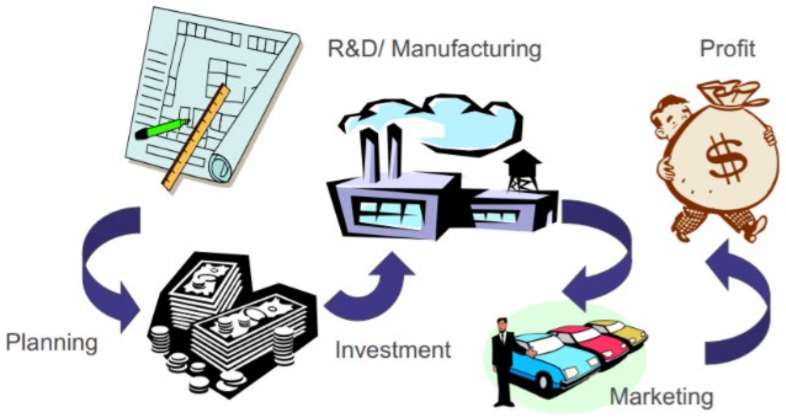
- Evaluates systematically the merits of proposed solutions to engineering problems.
- It balances trade-offs and select alternatives in the most economical way.
- It assists decision making based on economics
- Time and uncertainty are the defining aspects of any investment project.

For example, choosing the best design, recommending the best possible way to design/build a system, determining the optimal assignment/investments/capital allocations.

Why is it Important?

Engineering decisions have direct implications on costs.

How you engineer a product will depend on all of these stages:



Examples

Sometimes doing things in an economic fashion can inspire innovation and creativity. The two examples below show how finances can have an effect on engineering of products and services.

1. **Equipment & Process Selection.**

For example, choosing a panel for an automotive body. Choosing between plastic sheet moulding or steel sheet stock. The choice of material will dictate manufacturing process and costs:

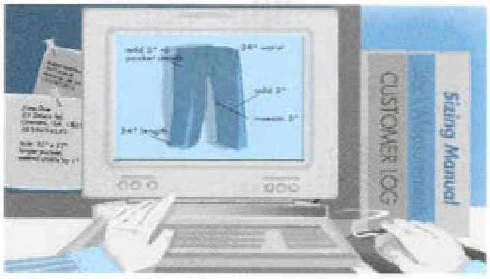
Description	Plastic SMC		Steel Sheet Stock
Material Cost ($/kg)	1.65	>	0.77
Machinery Investment	$2.1 million	<	$24.2 million
Tooling Investment	$0.683 million	<	$4 million
Cycle time (minute/part)	2.0	>	0.1

Other questions could include, is it worth spending on upgrading to a new facility, is it worth spending money to market a product.

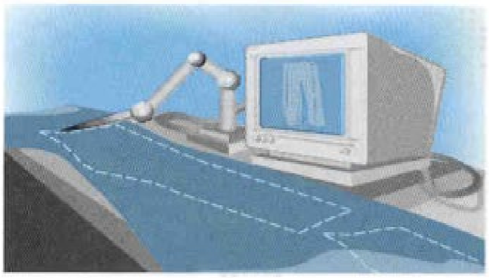
2. **Service Improvement**

Cost reduction questions could include should a company buy equipment to perform an operation done manually. And should money be spent now to save more later. E.g. Levi's:

A sales clerk measures the customer using instructions from a computer as an aid.



The clerk enters the measurements and adjusts the data based on the customer's reaction to samples.



The final measurements are relayed to a computerized fabric cutting machine at the factory.

Bar codes are attached to the clothing to track it as it is assembled, washed, and prepared for shipment.

Engineering Economy Principles

There is a rational decision-making process in a *pure engineering* setting:

1. Recognise decision problem – e.g. expanding capacity, marketing product better, reducing cost
2. Define goals and objectives
3. Collect all relevant information
4. Identify set of feasible alternatives – This is where the innovation can happen
5. Select decision criterion to use – what are the metrics that you decide performance
6. Select best alternative – Then apply tools to find this

The basic principles of the *engineering economy* are similar to this:

1. Develop the alternatives
2. Focus on the differences
3. Use a consistent viewpoint
4. Use a common unit of measure
5. Consider all relevant criteria
6. Make uncertainty explicit
7. Revisit your decision

Example Application of Principles

Using this idea, we can apply it to a case study of low-cost cars in India:

1. **Develop the alternatives**
   Carefully define the problem:
   - Car not affordable for all
   - Therefore, design a car to sell for £1000

   The final choice decision is along alternatives
   - Designs A, B, C … etc

   Alternatives need to be identified and then defined for subsequent analysis.

2. **Focus on the differences**
   Only differences in expected future outcomes are relevant to comparisons:
   - Design A: under budget, max speed 50kmh/h
   - Design B: on budget, max speed 60kmh/h
   - Design C: over budget, max speed 70kmh/h
   - *Design B': on budget, max speed 65kmh/h*

Only those should be considered in decisions

3. **Use a consistent viewpoint**
   Prospective outcomes of the alternatives, economics and , should be consistently developed from a defined viewpoint
   - Perspective of owner v perspective of customer

4. **Use a common unit of measure**
   Makes analysis easer and comparisons of alternative. If this is not possible, describe consequences explicitly to decision makers e.g. carbon footprint.

5. **Consider all relevant criteria**
   In decision making, selecting preferred alternatives relies on a criterion:
   - Cost
   - Speed
   - CO2 emissions
   - Etc

   Decision process should consider outcomes enumerated in monetary units and other metrics (e.g. patients treated)

6. **Make uncertainty explicit**
   Uncertainty affects future outcomes:
   - Future demand
   - Oil price
   - Emission standard regulations

   It should be recognised in analysis and comparison as early as possible. *This is important and described in more detail later in the notes.*

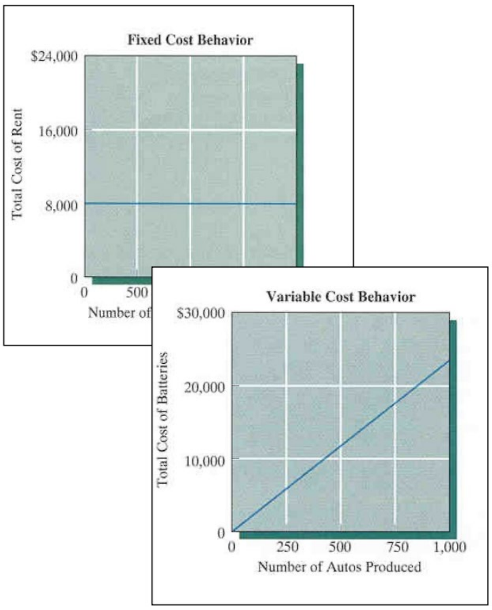
7. **Revisit your decision**
   Improved decision-making results from an adaptive process (i.e. flexibility).
   The initial projected outcomes of selected alternatives should be compared with actual results achieved. Then the decision/operation can be updated and modified.

L2. Costs

Common Cost Categories

There are some key cost types that are important. The figure below visualises them:

Fixed Costs	Variable Costs	Incremental Costs
• Unaffected by changes in activity level • E.g. insurance, taxes, general management and administration	• Vary in total with the quantity of output • E.g. materials, labour	• Additional costs from increasing output of a system by one or more units • E.g. cost of producing one more oil barrel, educating one more student



Other Cost Categories

As well as the 3 key cost types, some costs can be easier to allocate to specific activity:

Direct	Indirect	Standard Cost
• Can be measured and allocated to a specific work activity • E.g. labour, material	• Difficult to attribute or allocate to a specific output or work activity (also called overhead or burden) • E.g. common tools, electricity, property taxes, supervision	• Cost per unit output, established in advance of production or service delivery • E.g. estimating future manufacturing cost, preparing bids on products and/or services

Common Cost Terminology:

Cash cost:

- Involves a payment of cash

Book cost:

- Does not involve a cash transaction but is reflected in accounting stsem
- E.g. depreciation

Sunk cost:

- Has occurred in the past and has no relevance to estimates of future costs and revenues related to alternative course of action
- E.g. down payment or deposit

Opportunity cost

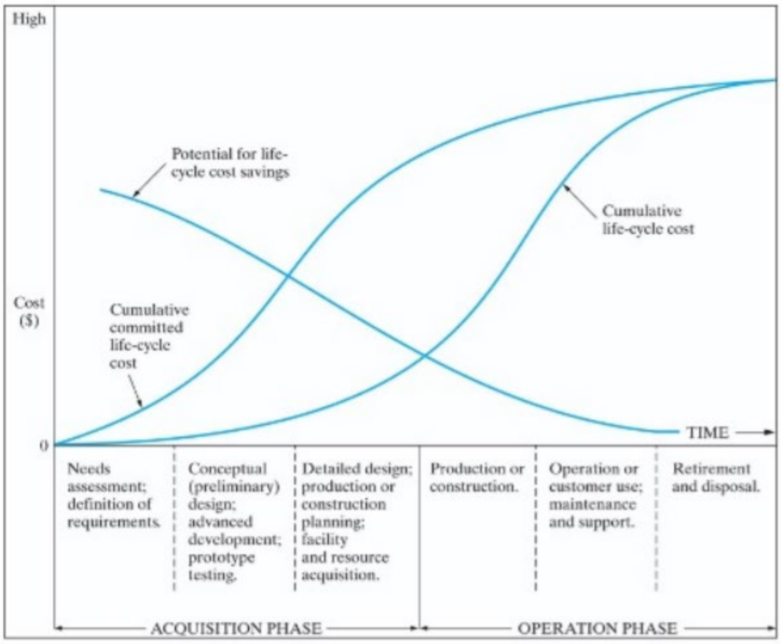
- Monetary advantage foregone due to limited resources
- The cost of the next best (rejected) opportunity
- E.g. the opportunity cost of being a full time student

Life-cycle cost

- Summation of all costs related to a product, structure, system, or service during its lifespan

There are two main phases of life-cycle cost as seen in the graph.

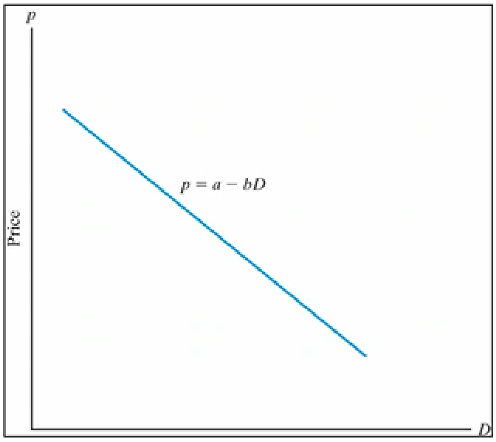
1. The *acquisition phase* is where you assess the need, define requirements, then go through conceptual design study and then move onto detail design of the system.
2. As time goes on in the *operations phase*, you commit these costs which accumulate, and as you spend them the cumulative life-cycle costs increases.



The takeaway from this is that the more time invested in the product, the less opportunity to save on the costs, which is intuitive.

General Price-Demand Relationship

There is a key relationship between the demand for something, and the price of it. This represents the notion of scarcity, and the more something is available out there the less you are willing to pay for it, e.g. gold.



Demand isn't the amount of gold people want, it's the scarcity of it.

- p: price
- D: demand (scarcity)
- a: y-intercept, maximum price at D=0
- b: slope (this is not always a straight line)

To find a and b, studies can be carried out, and then analysis such as regression can be used to find the constants.

Total Revenue

Total revenue (TR) is product of selling price per unit, p, and number of units sold, D.

$$\boldsymbol{p = a - bD}$$

$$TR = pD = (a - bD)D = aD - bD^2$$

$$for \quad 0 \leq D \leq \frac{a}{b} \qquad and \qquad a > 0, \;\; b > 0$$

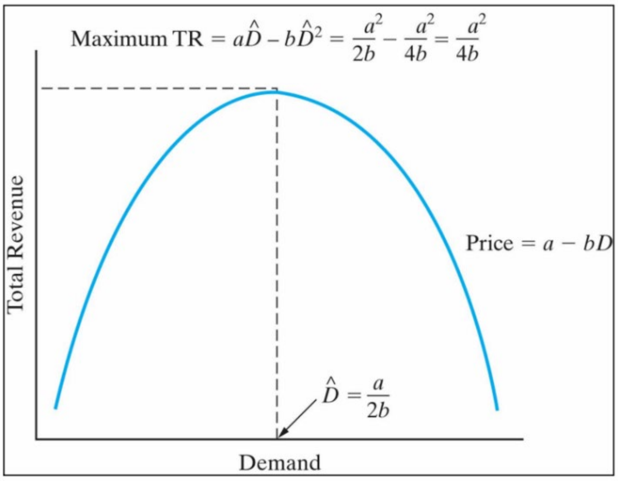
Optimal Volume/Demand for Revenue

By using the FONC from optimisation, we can take the derivative from the equation above to find the *optimal volume*:

$$\frac{dTR}{dD} = a - 2bD = 0$$

Solving optimal demand/volume:

$$\hat{D} = \frac{a}{2b}$$



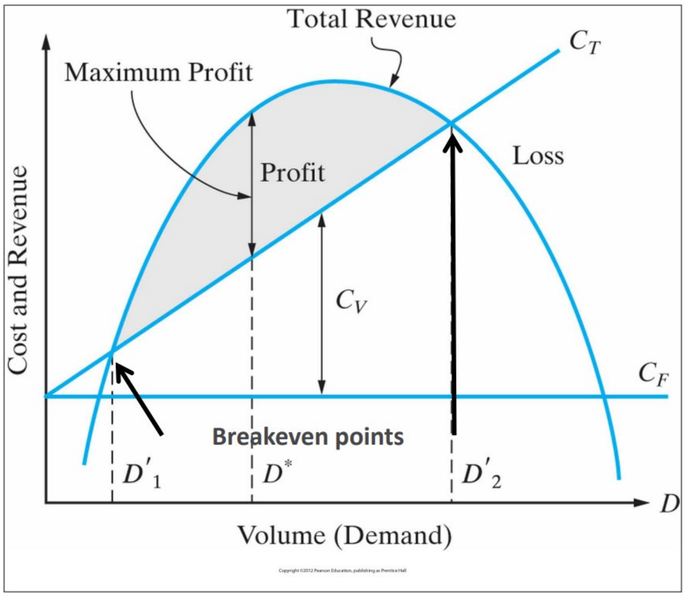
Profit can be Maximised

The graph below shows the relationships between revenues and costs. The curve as previously discussed, is the total revenue.

C_F is the fixed costs and is a horizontal line as they don't depend on the demand.

C_T are the total costs (fixed +variable), and the vertical distance between C_T and C_F is the variable cost C_V.

Profit is the total revenues - total costs, and hence D_1 and D_2 are breakeven points, where revenue equals costs.



Maximum profit at D^* occurs where there is the greatest difference between revenue and total costs. This is in between the two breakeven points.

Using the equation from earlier, we know that total revenue (TR) is:

$$TR = (a - bD)D = aD - bD^2$$

Total costs are the sum of variable and fixed costs:

$$TC = C_F + c_v D$$

Therefore, profit is:

$$TR - TC = aD - bD^2 - C_F - c_v D = -bD^2 + (a - c_v)D - C_F$$

This implies that *maximum profit* can be calculated by:

$$\frac{d(profit)}{dD} = a - c_v - 2bD = 0$$

$$\boldsymbol{D^* = \frac{a - c_v}{2b}}$$

Breakeven Analysis

As shown in the graph, there were two breakeven points. This is when costs = revenue:

$$TR = TC$$

$$aD - bD^2 = C_F + c_v D$$

$$-bD^2 + (a - c_v)D - C_F = 0$$

Finding the quadratic roots:

$$D' = \frac{-(a - c_v) \pm \sqrt{(a - c_v)^2 - 4(-b)(-C_F)}}{2(-b)}$$

Example of Revenue, Profit and Breakeven Calculations

A company produces an electronic timing switch used in consumer products. The fixed cost C_F = \$73000/month, and variable costs C_v = \$83 per unit. The selling price per unit is p = \$180 – 0.02(D). Determine optimal volume, and profit at this demand:

$$D^* = \frac{a - c_v}{2b} = \frac{\$180 - \$83}{2(0.02)} = 2{,}425 \text{ units per month [from Equation (2-10)].}$$

Is $(a - c_v) > 0$?

$$(\$180 - \$83) = \$97, \quad \text{which is greater than 0.}$$

And is (total revenue – total cost) > 0 for $D^* = 2{,}425$ units per month?

$$[\$180(2{,}425) - 0.02(2{,}425)^2] - [\$73{,}000 + \$83(2{,}425)] = \$44{,}612$$

A demand of $D^* = 2{,}425$ units per month results in a maximum profit of \$44,612 per month. Notice that the second derivative is negative (–0.04).

Find when breakeven occurs:

$$-bD^2 + (a - c_v)D - C_F = 0$$

$$-0.02D^2 + (\$180 - \$83)D - \$73{,}000 = 0$$

$$-0.02D^2 + 97D - 73{,}000 = 0$$

$$D' = \frac{-97 \pm [(97)^2 - 4(-0.02)(-73{,}000)]^{0.5}}{2(-0.02)}$$

$$D'_1 = \frac{-97 + 59.74}{-0.04} = 932 \text{ units per month}$$

$$D'_2 = \frac{-97 - 59.74}{-0.04} = 3{,}918 \text{ units per month.}$$

Thus, the range of profitable demand is 932–3,918 units per month.

Recall that a-c_v must be positive for D^* to be plausible (cannot have negative or even 0 optimal demand). SONC (from optimisation module) being negative confirms we have a global optimum.

Cost-Driven Design Optimisation

Engineers must consider cost design of products, processes, and services.

Cost-driven design optimisation' is critical in today's competitive business environment.

Let's consider discrete and continuous problems with single primary cost driver.

The approach is that:

- Design variable that is the primary cost driver is identified
- Express the cost model in terms of the design variable
- Optimise:
  - For continuous cost functions, differentiate to find optimal value
  - For discrete functions, calculate cost over a range of values of design variables e.g. using spreadsheet
  - Or if the solution is complex, use non-gradient algorithm e.g. particle swarm
- Find the optimum value, the minimum cost value

Simplified Cost Function

Here we introduce a new standard concept, which is a simplified cost function.

$$\boldsymbol{Cost = aX + \frac{b}{X} + k}$$

With this you can express any cost relationship as it captures directly varying costs, a, indirectly varying costs, b, and fixed costs, k.

This can also be built from first principles if you know fixed costs, variable costs, and capital costs.

Present Economy Studies

To end this chapter, it important to make a note of present economy studies. This is when the *'time value'* of money can be ignored, as alternatives are being compared over one year or less.

When revenues or other economic benefits *vary* among alternatives, choose the alternative maximizing overall profitability and defect-free output.

When revenues and other economics benefits *are not present or are constant* among alternatives, choose alternatives that minimises total cost per defect-free unit.

L3. Accounting

The objective of most companies includes increasing the market value of the company. The market value is the stock price reflected in the financial market.

The market values of some of the well-known U.S. firms are shown below as of 2022:

Company	Stock Price ($)	Volume trading	Market Cap (B$)
Apple	175.53	74,805,173	2,867,000
Facebook	75.18	24,494,756	210,245
Google	2,832.96	1,182,079	1,878,000

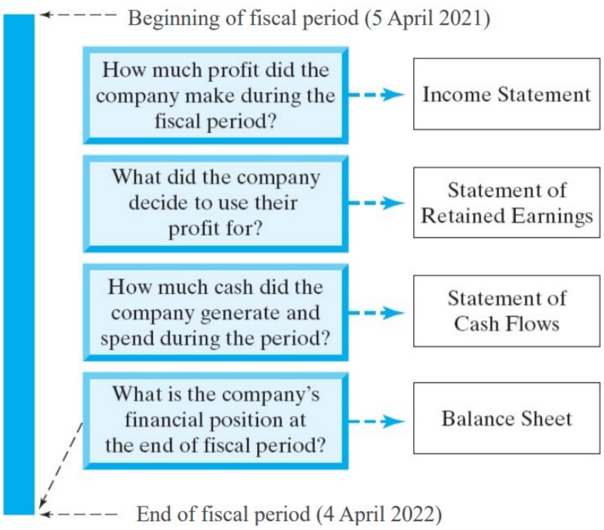
Factors affecting market value include:

- How well a company is doing at a particular time
- What is happening to other stock prices, how are competitors doing in that sector
- How so investors expect the company to perform in the future

Engineers should understand financial statements as engineering is one component of a complex decision-making and valuation process. The role of engineers is to:

- Evaluate capital expenditure delated to projects
- Selection of production methods used
- Assessment of engineering safety and environmental impact
- Selection of types of products or services used

There are different types of financial status for businesses:



Balance Sheet

The *balance sheet* gives a quick ‘snapshot of time’ on company’s financial situation.

It reports on assets, liabilities, stockholders’ equity. Assets are typically arranged in order of liquidity. Liabilities are typically arranged in order of payments.

For balance sheet presentation this MUST be true:

Assets = Liabilities + Owners’ Equity

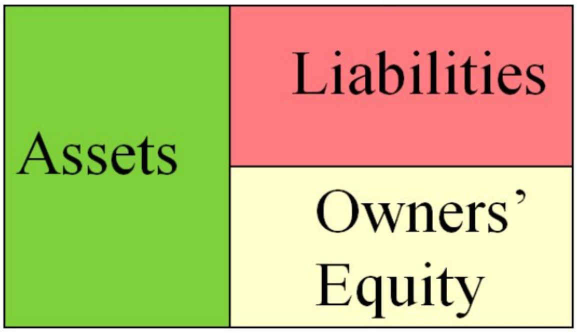
For the financial analysis:

Assets - Liabilities = Owners’ Equity

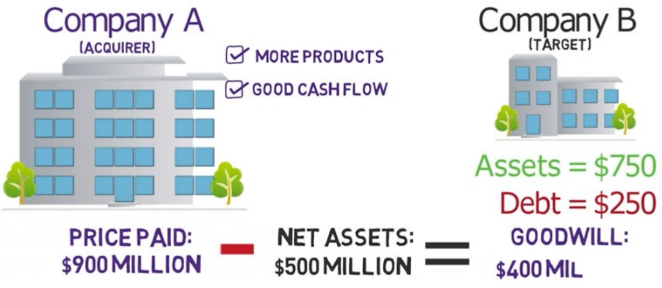
Here is an example of a complete balance sheet. It can be seen that the assets total equals the liabilities plus the shareholders equity.

		As at February 28, 2009	As at March 1, 2008
Assets			
Current			
Cash and cash equivalents		$ 835,546	$ 1,184,398
Short-term investments	< 1 year	682,666	420,709
Trade receivables	e.g. account receivables	2,112,117	1,174,692
Other receivables		157,728	74,689
Inventory	e.g. BlackBerry stocks	682,400	396,267
Other current assets		187,257	135,849
Deferred income tax asset		183,872	90,750
		4,841,586	3,477,354
Long-term investments		720,635	738,889
Capital assets	e.g. land, buildings, machinery	1,334,648	705,955
Intangible assets	e.g. copyrights, franchise	1,066,527	469,988
Goodwill		137,572	114,455
Deferred income tax asset		404	4,546
		$ 8,101,372	$ 5,511,187
Liabilities			
Current			
Accounts payable	< 1 year, e.g. notes payable	$ 448,339	$ 271,076
Accrued liabilities	e.g. wages, salaries, rent	1,238,602	690,442
Income taxes payable		361,460	475,328
Deferred revenue	goods/services not yet delivered	53,834	37,236
Deferred income tax liability		13,116	—
Current portion of long-term debt	interests on debt	—	349
		2,115,351	1,474,431
Deferred income tax liability		87,917	65,058
Income taxes payable		23,976	30,873
Long-term debt		—	7,259
		2,227,244	1,577,621

		As at February 28, 2009	As at March 1, 2008
Shareholders’ Equity			
Capital stock			
Common shares Issued - 566,218,819 voting common shares (March 1, 2008 - 562,652,461)		2,208,235	2,169,856
Retained earnings		3,545,710	1,653,094
Additional paid-in capital	additional stock sales	119,726	80,333
Accumulated other comprehensive income		457	30,283
		5,874,128	3,933,566
		$ 8,101,372	$ 5,511,187



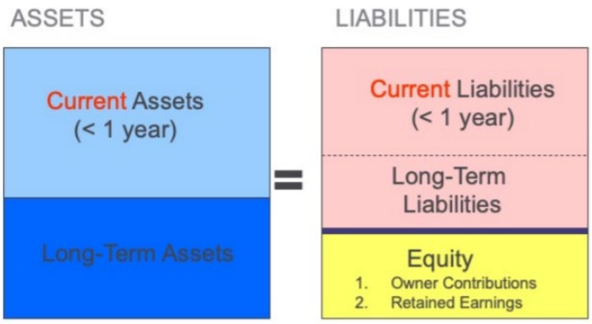
Note that *goodwill* arises when one company acquires another company but pays more than the fair market value of the net assets (total assets – total liabilities). It is classified as an intangible asset on the balance sheet.



A firm gets equity in two ways:

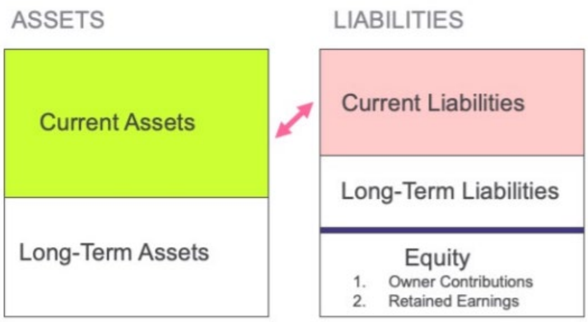
- *Owners' contributions*: by issuing stocks through financial markets, and initial amounts contributed
- *Retained earnings*: by retaining operating profits instead of paying dividends.

There are *four key quadrants* of the balance sheet.

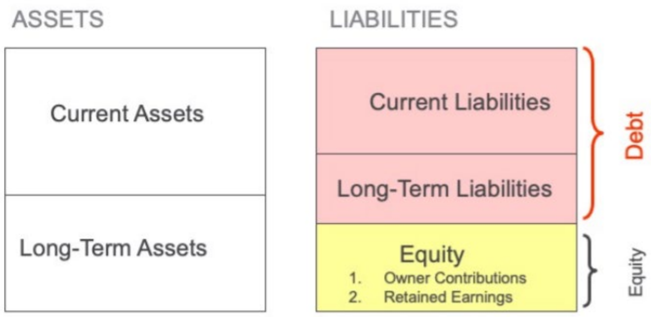


These can be used to calculate some ratios that are used in financial analysis:

1. **Current ratio** is the current assets/current liabilities. The current ratio is an indication of a firm's *market liquidity* and ability to meet creditors demand. Acceptable current ratios may vary from industry to industry and are generally between 1.5 and 3 for healthy businesses.



2. **Debt to equity ratio** is a ratio indicating the relative proportion of the shareholders equity and debt used to finance a company's assets.



Income Statement

The **income statement** indicates whether a company is making or losing money over a stated period (usually a year). They can be used as profit check points.

The basic income statement equation is:

The **gross margin** is a critical measure for engineers:

COGS can be reduced through smart engineering, by reducing costs of tech, inventory management and better processes etc….

Cost of good sold refers to the carrying value of goods sold during a particular period. Costs are associated with particular goods using one of several formulas, including specific identification, first-in first-out, last-in first-out, or average cost.

Costs include all costs of purchase, costs of conversion and other costs incurred in bringing inventories to their present location and condition.

The costs of those goods not sold are deferred as costs of inventory until the inventory is sold or written down in value.

Calculating the Cost of Goods Sold

Beginning Inventory

+ Additions to Inventory

- Ending Inventory

Cost of Goods Sold

For example, starting with an inventory of 10 million phones, and adding 2 million phones, but you end the year with 8 million phones in the inventory. This means you have sold 10+2-8= 4 million phones. COGS is the number of items sold x assumed sales price. Suppose a phone is $400 to produce, shop and maintain on average, then COGS = 4mil x $400 = $1.6 billion.

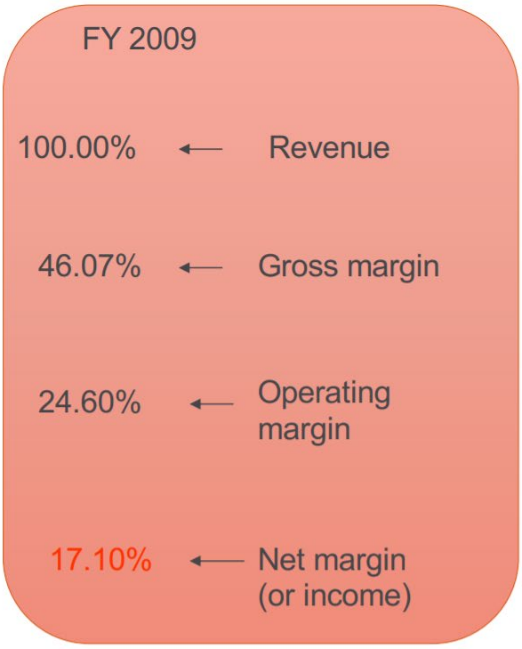
There are number of places for **profit check points.** These can be thought of as percentage margins of the total revenue:

ABC Company, Inc. Statement of Operations (Year Ended December 31, 20xx)		
Sales	$5,000,000	100.0%
Less: Cost of Goods Sold	3,250,000	65.0%
Gross Profit (margin)	1,750,000	35.0%
Less: Selling, G&A Expenses	1,000,000	20.0%
Operating Profit (margin)	750,000	15.0%
Less: Interests	250,000	5.0%
Net Income Before Taxes (NBIT)	500,000	10.0%
Less: Taxes	175,000	3.5%
Net Income (margin)	$325,000	6.5%

Bottom line

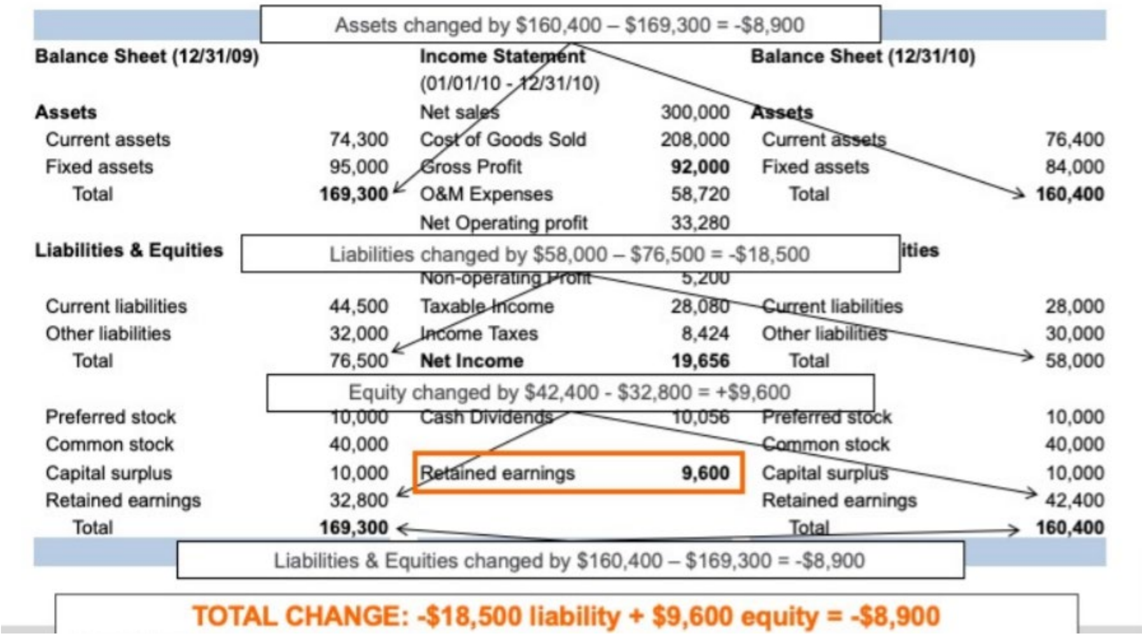
If we now look at an example income statement of RIM Ltd, the revenue can be used to see the percentage of the margins of the revenue:

	For the Year Ended February 28, 2009	March 1, 2008	March 3, 2007
Revenue			
Devices and other	$ 9,410,755	$ 4,914,366	$ 2,303,800
Service and software	1,654,431	1,095,029	733,303
	11,065,186	6,009,395	3,037,103
Cost of sales			
Devices and other	5,718,041	2,758,250	1,265,251
Service and software	249,847	170,564	114,050
	5,967,888	2,928,814	1,379,301
Gross Margin	5,097,298	3,080,581	1,657,802
Expenses			
Research and development	684,702	359,828	236,173
Selling, marketing and administration	1,495,697	881,482	537,922
Amortization	194,803	108,112	76,879
	2,375,202	1,349,422	850,974
Income from operations	2,722,096	1,731,159	806,828
Investment income	78,267	79,361	52,117
Income before income taxes	2,800,363	1,810,520	858,945
Provision for (recovery of) income taxes			
Current	948,536	587,845	123,553
Deferred	(40,789)	(71,192)	103,820
	907,747	516,653	227,373
Net income	$ 1,892,616	$ 1,293,867	$ 631,572
Earnings per share			
Basic	$ 3.35	$ 2.31	$ 1.14
Diluted	$ 3.30	$ 2.26	$ 1.10



Balance Sheet & Income Statement Relationship

There is a relationship between balance sheets and income statement. If we looked at two balance sheets a year apart, and the income statement between these two points we can see that the retained earnings (8900) on the balance sheet equals the increase in retained earnings on the income statement.



Note that **preferred stock**, also called preferred shares, preference shares or simply preferred, is a stock which may have a combination of features not possessed by common stock including properties of both an equity and debt instrument and is generally considered a hybrid instrument.

Preferred are senior to common stock, but subordinate to bonds in terms of claim (or rights to their share of the assets of the company) and may have priority over common stock in the payment of dividends and upon liquidation.

Cash Flow Statement

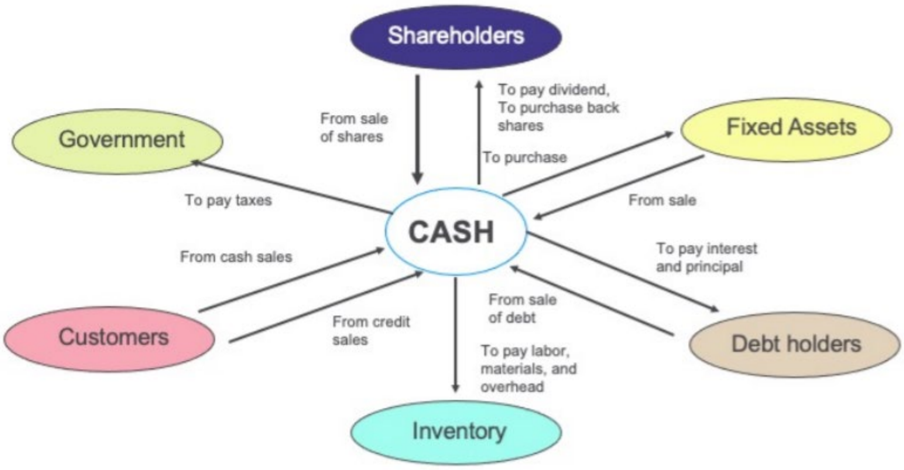
So far, we have seen that:

- The balance sheet gives us a financial snapshot in time about a company
- An income statement indicates whether a company is making or losing money over a time period (usually a year).

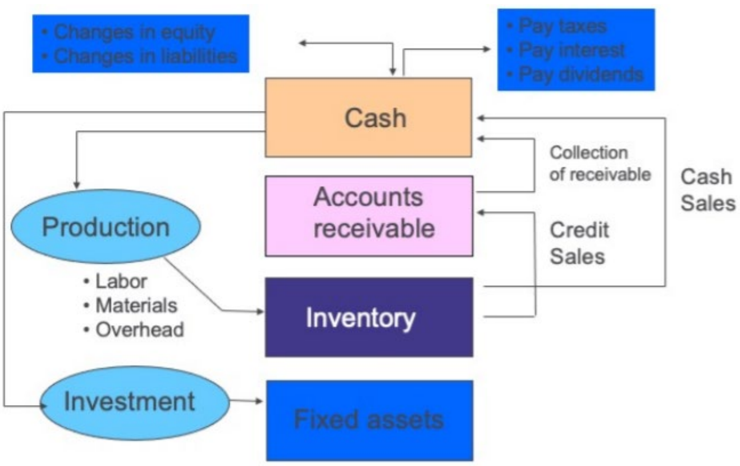
Now, a statement of *cash flows* shows how the company:

- Generates the cash it receives (sources)
- Uses the cash over the reporting period

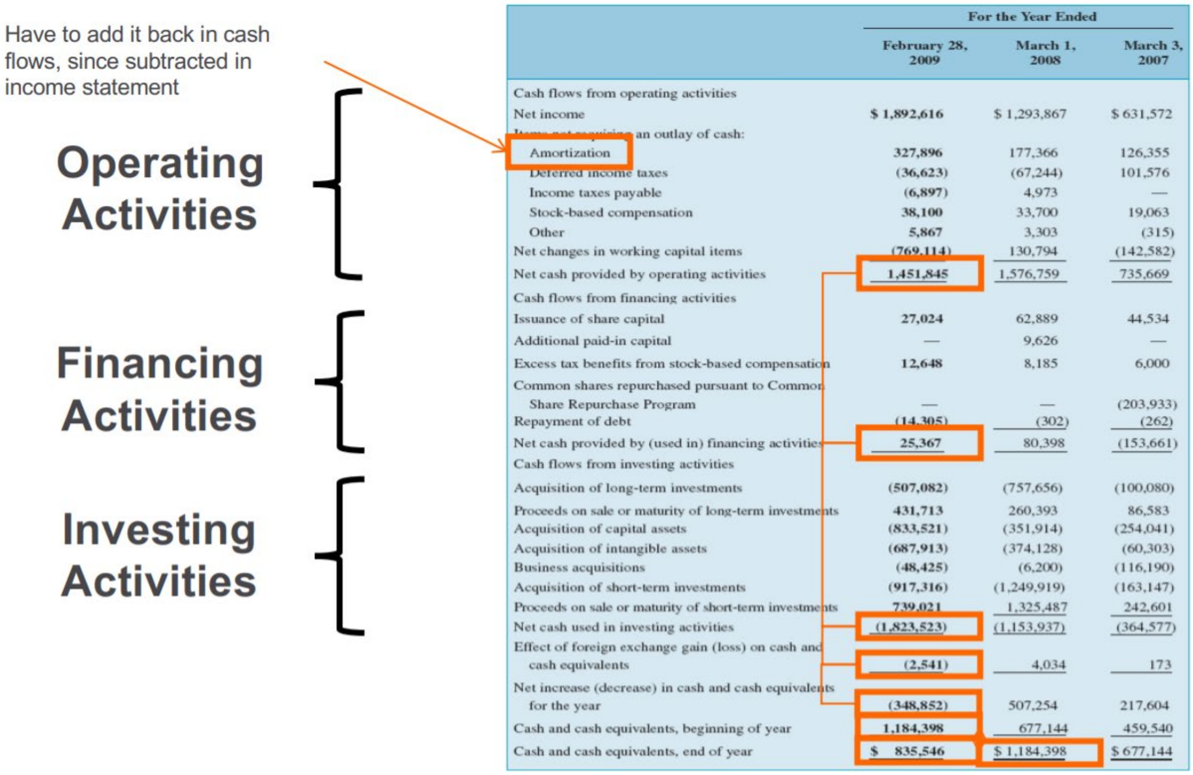
There are many flows of cash transactions within a business:



This shows a flow more from an accounting perspective:

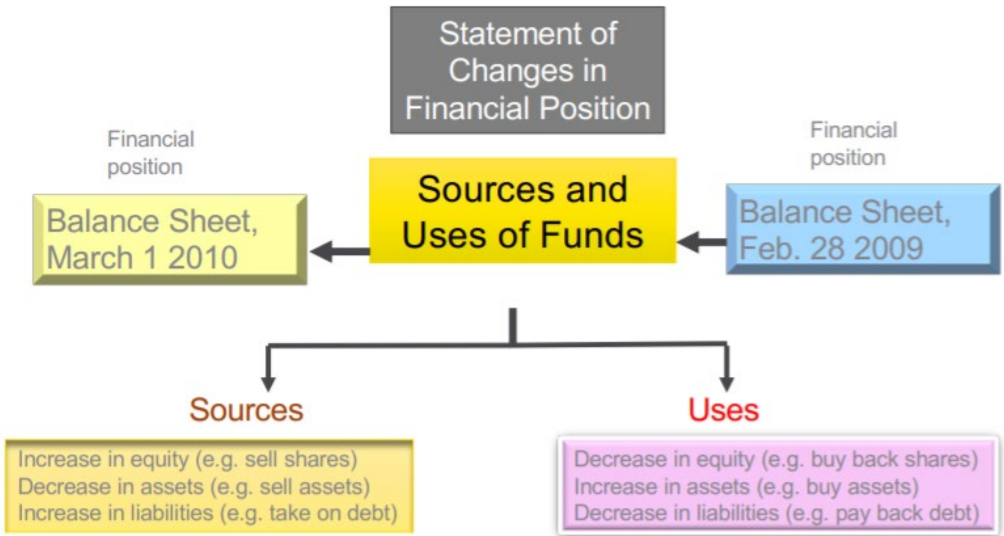


Here is an example of a cash flow statement from RIM Ltd, and these cash flows can be split into 3 types of activities:



	For the Year Ended		
	February 28, 2009	March 1, 2008	March 3, 2007
Cash flows from operating activities			
Net income	$ 1,892,616	$ 1,293,867	$ 631,572
Items not requiring an outlay of cash:			
Amortization	327,896	177,366	126,355
Deferred income taxes	(36,623)	(67,244)	101,576
Income taxes payable	(6,897)	4,973	—
Stock-based compensation	38,100	33,700	19,063
Other	5,867	3,303	(315)
Net changes in working capital items	(769,114)	130,794	(142,582)
Net cash provided by operating activities	1,451,845	1,576,759	735,669
Cash flows from financing activities			
Issuance of share capital	27,024	62,889	44,534
Additional paid-in capital	—	9,626	—
Excess tax benefits from stock-based compensation	12,648	8,185	6,000
Common shares repurchased pursuant to Common Share Repurchase Program	—	—	(203,933)
Repayment of debt	(14,305)	(302)	(262)
Net cash provided by (used in) financing activities	25,367	80,398	(153,661)
Cash flows from investing activities			
Acquisition of long-term investments	(507,082)	(757,656)	(100,080)
Proceeds on sale or maturity of long-term investments	431,713	260,393	86,583
Acquisition of capital assets	(833,521)	(351,914)	(254,041)
Acquisition of intangible assets	(687,913)	(374,128)	(60,303)
Business acquisitions	(48,425)	(6,200)	(116,190)
Acquisition of short-term investments	(917,316)	(1,249,919)	(163,147)
Proceeds on sale or maturity of short-term investments	739,021	1,325,487	242,601
Net cash used in investing activities	(1,823,523)	(1,153,937)	(364,577)
Effect of foreign exchange gain (loss) on cash and cash equivalents	(2,541)	4,034	173
Net increase (decrease) in cash and cash equivalents for the year	(348,852)	507,254	217,604
Cash and cash equivalents, beginning of year	1,184,398	677,144	459,540
Cash and cash equivalents, end of year	$ 835,546	$ 1,184,398	$ 677,144

There are different types of sources and uses of cash. The financial position at the end of one fiscal year, considering all the sources and uses of cash that happen during that year will lead you to the balance sheet at the end of that particular fiscal year:



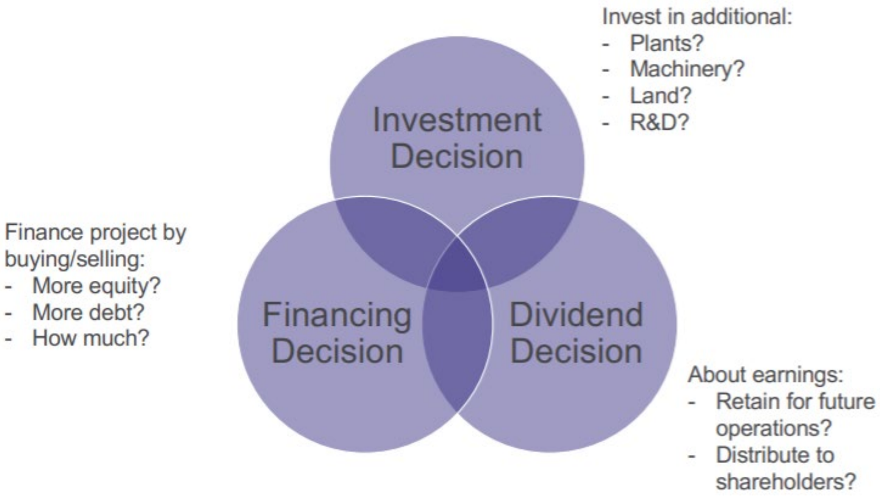
Capital Budgeting

Capital is fixed assets used in production.

Budget is a plan which details projected cash flows during some future period

Capital budgeting is a process of analysing projects and deciding whether they should be included in the capital budget, and how to finance them.

There is a capital budgeting **framework** for this allocating of finances**:**



In this there are 3 areas of decisions, so what to invest in, how should we finance these projects, and what should we do with earnings.

Ratio Analysis

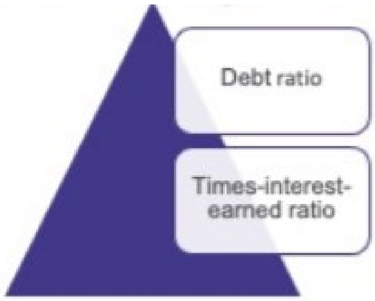
There are some key *financial ratios* used to evaluate companies performance and health. They can include**:**

- Debt management
- Liquidity
- Asset management
- Profitability
- Market Trend/Value
- Trends and graphs to spot problems

We will investigate some specific ones below.

1. Debt management is the first form:

- These are ratios to show how a firm uses debt financing and its ability to meet debt repayment obligations.
- They are of particular interest to creditors



Debt Ratio formula is:

$$\text{Debt ratio}=\frac{\text{Total debt}}{\text{Total assets}}$$

(total debt is actually total liabilities)

Times interest earned measures the extent to which earnings can decline without defaulting on debt service.

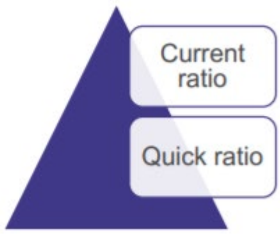
This is also known *as interest coverage ratio* (EBIT=earnings before interest and taxes).

.

$$\text{Times Interest Earned} = \frac{\text{EBIT}}{\text{Interest Charges}}$$

2. Liquidity analysis are ratios showing the relationship of a firms cash and other assets to its current liabilities. They are of interest mainly to suppliers and short-term creditors.
Current ratio:

$$\text{Current Ratio} = \frac{\text{Current Assets}}{\text{Current Liabilities}}$$



This is best rule of thumb for determining the firm's liquidity.
The rule of thumb is 2:1.

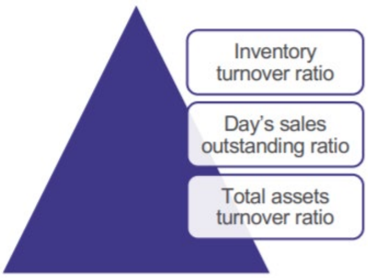
A *quick ratio* says whether a company could pay all current liabilities immediately. It excludes inventories and prepaid expenses:

$$\text{Quick Ratio} = \frac{\text{Current Assets - Inventories}}{\text{Current Liabilities}}$$

Liquidity ratio is an indication of a firm's immediate liquidity. The formula is:

$$\text{Liquidity Ratio} = \frac{\text{Cash+Cash Equ.}}{\text{Current Liabilities}}$$

3. Asset management analysis includes a set of ratios measuring how effectively a firm is managing its assets.



The *industry turnover ratio* highlights the rate at which the inventory is being sold:

$$\text{Inventory Turnover} = \frac{\text{Cost of Goods Sold}}{\text{Average Inventory}}$$

Days sales outstanding (DSO) determines the average length of time the firm waits before receiving cash payment.

$$\text{DSO} = \frac{\text{Account Receivable}}{\text{Average sales per day}}$$

Total Asset turnover measures how effectively the firm uses its total assets in generating its revenues.

$$\text{Total Asset Turnover} = \frac{\text{Net Sales}}{\text{Total Assets}}$$

4. Profitability analysis is a set of ratios which show the combined effects of liquidity, asset management, and debt on operating results. This is of interest to investors and owners (i.e. shareholders).

Gross margin indicates the profitability of the sales effort:

$$\text{Gross Margin Ratio} = \frac{\text{Gross Profit (\$)}}{\text{Net Sales}}$$

Net margin indicates the profit per dollar of sales:

$$\text{Net Margin Ratio} = \frac{\text{Net Income (\$)}}{\text{Net Sales}}$$

Return of equity (ROE) measures rate of return on owners investment (or common equity)

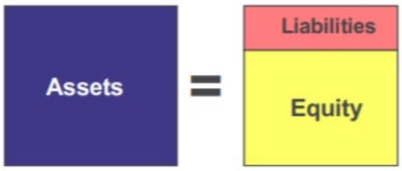
$$\text{Return on Equity} = \frac{\text{Net Income}}{\text{Average Common Equity}}$$

$$= \frac{\text{Net income}}{\text{Sales}} \times \frac{\text{Sales}}{\text{Assets}} \times \frac{\text{Assets}}{\text{Average shareholders' equity}}$$

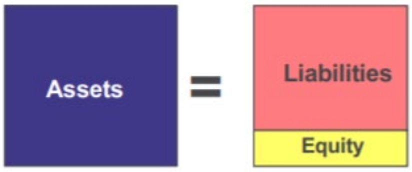
$$= (\text{Profit margin}) \times (\text{Asset turnover}) \times (\text{Financial leverage})$$

Look at the size of equity in the ROE denominator.

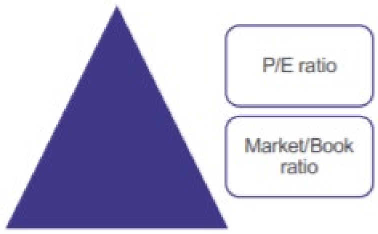
An example of a healthy company that might not have a spectacular ROE because there is so much equity in the company (e.g. well established DOW 30 stocks)



An example of a highly leveraged company that might have spectacular ROE because owners have put little of own resource into company e.g. high-tech industries)



5. Market trend analysis is a set of ratios that relate the firms stock price to its earnings and book value per share. This is of particular interest to investors and shareholders (owners).



Earnings per share indicates the earning attributable to each share of stock.

It is a widely used indicator of a corporation's performance.

$$\text{EPS} = \frac{\text{Net Income}}{\text{Common Shares Outstanding}}$$

Price to earnings ratio (P/E) indicates how much investors are willing to pay per dollar of reported profits.

$$\text{P/E ratio (2009)} = \frac{\text{Price per share}}{\text{EPS}}$$

As part of stock analysis, you need to consider what premium you are willing to pay for a company's earnings today and determine if the expected growth warrants the premium.

Also use comparables to see relative valuation and determine whether the premium is work the cost.

Book value/share indicates what the value of a share of stock is according to the books financial statements).

$$\text{Book Value/Share} = \frac{\text{Equity - Preferred stock}}{\text{Average Shares Outstanding}}$$

Trends analysis is where one plots a ratio over time. It is important because it reveals whether the firm's ratios are improving or deteriorating over time. E.g. changing interest coverage ratio may indicate higher/lower use of leverage, and impact prospects for growth.

There are **limitations** of financial ratios.

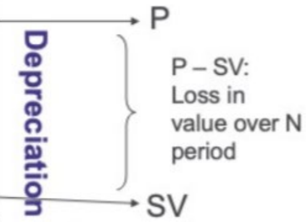
- Ratio analysis is useful, but analysts should be aware of ever-changing market conditions and make necessary adjustments
- It is difficult to generalise whether a particular ratio is good or bad, as comparisons with industry peers should always be used.
- Ratio analysis based on any one year may not represent the true business condition, like driving forward using a rear-view mirror.

L4. Depreciation Taxes

Depreciation Introduction

The definition of depreciation is *the loss of value for a fixed asset over time*. For example, purchasing a production equipment worth £15,000 at the beginning of the year 2022:

End of Year	Market Value	Loss of Value
0	£15,000	
1	10,000	£5,000
2	8,000	2,000
3	6,000	2,000
4	5,000	1,000
5	4,000	1,000



This loss in value is essentially a cost of owning the asset, and hence must be properly accounted for.

Depreciation is an accounting concept, a **non-cash** cost that establishes an annual deduction against before-tax income. It is intended to approximate yearly fraction of an asset's value used in the production of income.

Depreciation is viewed as a **business expenses** reducing taxable income. By including depreciation, it helps to reduce your profit, and therefore mean you pay less taxes.

Gross Income - Expenses:
(Cost of goods sold)
(Depreciation)
(Operating expenses)

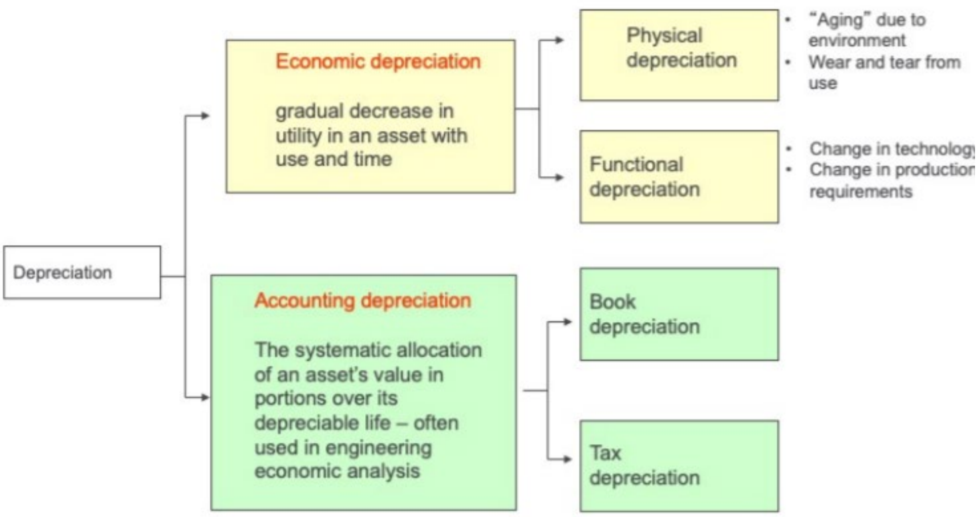
Taxable Income

- Income taxes

Net income

A government may allow a 100% depreciation in one year for some assets (meaning less taxes paid by company) in order to incentivise the purchase of it.

There are two types of depreciation. Economic (the 'real' decrease in an assets utility over time) and accounting (the financial side), and we will focus on the latter:



What can be depreciated?

- Assets used in business or held for production of income
- Assets having definite useful life and life longer than one year
- Assets that can wear out, become obsolete or lose value
- **Not inventory, stock in trade, or investment property**

Factors to consider in asset depreciation. These will be explored in this section:

- Depreciable life
- Salvage value
- Cost basis
- Method of depreciation

Depreciation Definitions

Basis or cost basis (B): The initial cost of acquiring an asset (purchase price + any sale taxes), including transportation, installation, start up cost.

Cost of new hole-punching machine (Invoice price)	£62,500
+ Freight	725
+ Installation labor	2,150
+ Site preparation	3,500
Cost basis to use in depreciation calculation	£68,875

Book value (BV): Worth of a depreciable property as shown in the accounting record of a company. The book value at the end of the year *k* is the cost basis minus the total allowable depreciations up to that year, where d_i is depreciation allowed or claimed in year *i.*

$$BV_k = B - \sum_{i=1}^{k} d_i$$

Market Value (MV): The amount for which the property can be sold in open market. It may change from year to year. We denote the market value of an asset at the end of the year by MV_k.

Recovery Period (R): It is the number of years over which the cost basis of an asset or property is recovered through the accounting process. It's the number of years to make BV=0.

Useful Life (N): The expected or estimated period that a property will be used in a trade or business to produce income. Note that *N* need not be equal to N.

Salvage Value (SV): The estimated value of a property at the end of its useful life, *N.* This is denoted by SV_N.

Depreciation Methods

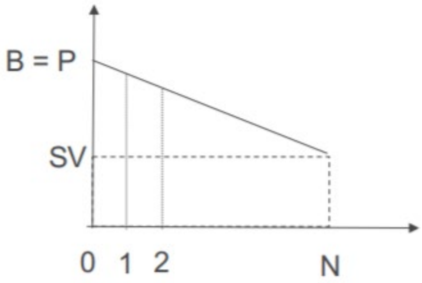
There are two different types of depreciation methods:

1. Straight Line Method, Declining Balance Method and Declining Balance Method with switch over to straight line are *Time-based methods.*
2. Unit Production Method is *Volume-based.*

We will look at these below.

1. Straight-line method assumes a constant amount is depreciated each year over the asset's life.

- B = Costs basis
- N – useful like of asset in years
- SV = estimated salvage value



Annual depreciation amount:

$$d_k = \frac{(B - SV)}{N}, and\ constant\ for\ all\ n$$

Cumulative depreciation through year k:

$$d_k^* = kd_k$$

Book value at end of year *k*:

$$BV_k = B - d_k * = B - kd_k$$

2. Declining balance method assumes annual depreciation is a constant percentage of the asset value at the beginning of the year.

It is an accelerated write-off technique, with the largest depreciation in the first year. Also known as the constant percentage method, or Matheson formula.

The depreciation amount for year 1: $d_1 = R \times B$, *where* $0 < R < 1$

The cumulative depreciation through year *k*, $d_k = B[1-(1-R)^k]$ due to the property of sun of n first terms of geometric series.

For example, a piece of equipment has cost basis B=£4000, N=10 years. Estimated SV at end of 10 years is 0. Use R=2/N. What is depreciation over 10 years? What is BV of equipment at end of 10 years.

R=2/N = 2/10 = 0.2

So 20% less from previous value. Note that depreciation does not consider SV_N.

The rate of depreciation can be determined to make final book value = salvage value

$$BV_N = B(1-R)^N = SV_N$$

$$R = 1-[SV_N/B]^{1/N}$$

n	d_k	BV_k
0	-	4000.00
1	800.00	3200.00
2	640.00	2560.00
3	512.00	2048.00
4	409.60	1638.40
5	327.68	1310.72
6	262.14	1048.58
7	209.72	838.86
8	167.77	671.09
9	134.22	536.87
10	107.37	429.50

For the declining balance method, is R is specific, there is no guarantee that BV will become SV at time N. Hence, switching from declining balance to another method, usually straight line is allowed.

In the year where straight line has larger depreciation than the declining method is the time to switch. The general principle is to use the method which gives the highest depreciation.

3. Units-of-production methods is different, as the previous depreciation methods are time based. Depreciation may be based on a method not expressed in terms of years if the decrease in value is mostly a function of use.

The cost basis (minus final SV) is allocated equally over the estimated number of units produced during the useful life of the asset:

$$Depreciation\ per\ unit\ of\ production = \frac{B - SV_N}{Estimated\ lifetime\ prouction\ in\ units}$$

Below is an example for the application of the units of production method. An equipment has B=£50,000 and an expected SV of £10,000 after 30,000 hours of use. Find its depreciation rate per hour of use, and its BV after 10,000 hours of operation.

$$= \frac{50{,}000 - 10{,}000}{30{,}000} = £1.33\ per\ hour\ of\ production$$

After 10,000 hours its book value:

$$BV_{10{,}000\ hours} = 50{,}000 - 1.33(10{,}000) = £36{,}700$$

Taxes

Taxes paid by a company is the cost of doing business. They are important since they represent a major cash flow, thus affecting the viability of an investment.

Taxes are based on a company's income; hence an after-tax analysis will reflect a more accurate measure of investment potential.

There are *different types* of taxes:

- Property taxes: function of value of real estate, business and personal property, and they are independent of income or profit.
- Sales taxes: Based on purchases of goods and services (VAT) and independent of income or profit
- Excise taxes: Function of sale of certain non-essential goods and independent of income or profit e.g. tobacco or liquor
- Income taxes: Function of gross revenues minus some allowable deductions, and the most important thing in engineering economic studies

Objectives of the tax policy include:

- *Revenue raising*, the traditional aims and is a substantial source of government funding
- *Promotion of economic goals*, as tax has been used to influence behaviour towards desirable social and economic goals e.g. tax discounts to stimulate a given economic sector

If we look at the UK specifically, we can see the traditional *tax rates*:

- 1976: 35%
- 2007: 20%
- 2010: 50%
- 2012: 45%

The tax *payable* for UK companies is given by:

$$Tax\ payable = Taxable\ income \times Tax\ \ rate$$

The taxable income includes *income inside and outside* the UK:

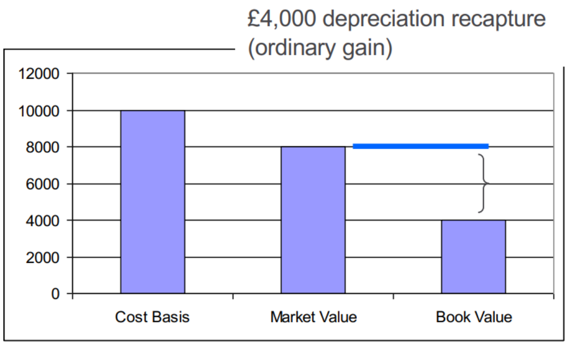
- Allowable deductions
- Capital allowances (i.e. depreciation of assets)
- Losses
- Approved donations

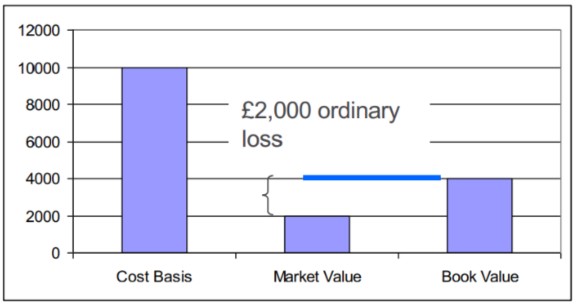
How Depreciation Affects Company Tax

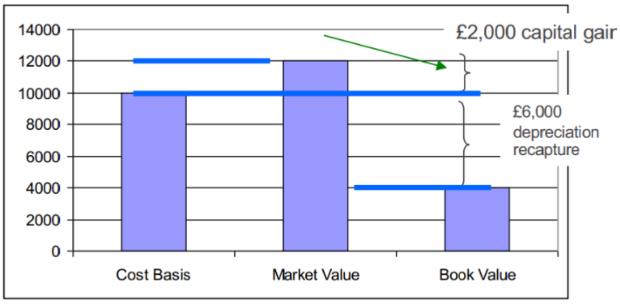
When an asset is disposed of for more (less) than its book value, the resulting gain (loss) is taxed, usually the same rate as for ordinary income or loss.

If market value is higher than its cost basis, a special tax may be levied on the profits.

Taxable capital gain is capped at the total capital allowances previously granted to the asset.







For example, an asset having a current book value of £5000 is sold for £8000. What tax liability is associated with this sale if the tax rate, t, is 20%.

$$Ordinary\ gain \ = \ £3000$$

$$Tax\ liability = t * disposal\ gain = 0.20(3000) = £600$$

After tax cash flow (ATCF) for sale of asset:

$$8000 - 600 \ = \ £7400$$

You are essentially being taxed for selling the asset at a higher value than its book value. This tax rule is the same for a loss, but the tax liability is now credit, so you get rewarded for selling it for less than book value.

ATCF are used in place of before-tax cash flows (BTCF) by including expenses (or savings) due to income taxes. Notation for this is as follows:

- R_k is revenues and savings from the project, this is cash inflow from the project during period
- E_k is cash outflows during year k for deductible expenses and interest
- d_k is sum of all non-cash or book costs during year k such as depreciation or depletion
- t is effective income tax on ordinary income; t is assumed to remain constant during the study period.
- T_k is the income taxes paid during year k.

Net income before tax for year k is:

$$NIBT_k = R_k - E_k - d_k$$

Income tax payable for year k is:

$$T_k = t(R_k - E_k - d_k)$$

Net income after tax for year k is:

$$NIAT_k = NIBT_k - T_k$$

$$= R_k - E_k - d_k - t(R_k - E_k - d_k)$$

$$= (1 - t)(R_k - E_k - d_k)$$

After tax cash flow for year k is the $NIAT_k$ plus non-cash items such as depreciations:

$$ATCF_k = NIAT_k - d_k$$

$$= (1 - t)(R_k - E_k - d_k) + d_k$$

$$= (1 - t)(R_k - E_k) + td_k$$

L5. Time Value

Money has value in time, it can fructify over time (e.g. in saving account or loans), and its value usually measured using an interest rate. That interest rate is the cost of borrowing money.

For example:

- Invest £1 for 1 year at 3%, and worth £1.03 next year
- Therefore, it will take £1 today to but £1.03 next year.
- Therefore £1 today it worth more than £1 next year.

Two Types of Interest

There is **simple interest**, interest charged on a principal amount (initial sum).

E.g. a principal amount (P=£1000), interest rate (i=8%), number of interest periods (N=3) and interest earnt (I).

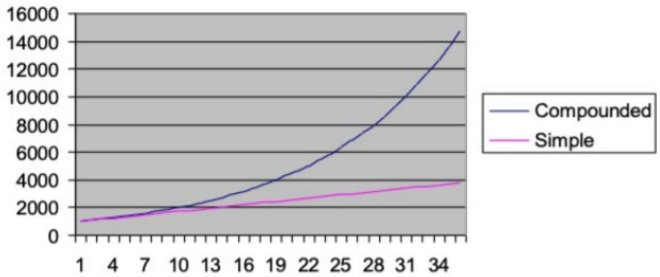
Using formula I=P x i x N = 1000 x 3 x 1.08 = £240

End of Year	Beginning Balance	Interest earned	Ending Balance
0			£1,000
1	£1,000	£80	£1,080
2	£1,080	£80	£1,160
3	£1,160	£80	£1,240

Compound interest is the practice of charging an interest rate to an initial sum and to any previously accumulated interest that has not been withdrawn.

End of Year	Beginning Balance	Interest earned	Ending Balance
0			£1,000
1	£1,000	£80	£1,080
2	£1,080	£86	£1,166
3	£1,166	£94	£1,260

This graph shows the effect of compound interest versus simple interest. If you take out a loan with a compound interest rate, you will have to pay back more compared to a simple interest rate:



Rule of 72 is a rule of thumb to be able to assess how long it takes to double an amount of money.

It is based on the Taylor series expansion, so the value of an asset s at a time n is equal to the value at time 0 multiplied by 1+the interest rate that you can gain to the power of that number of years:

$$S_n = S_0(1+i)^n \sim S_0 e^{in} \; by\; Taylor\; Series\; expansion$$

$$If\;\; S_n = 2S_0 \rightarrow 2S_0 = S_0 e^{in} \rightarrow \ln(2) = in = 0.693 \sim 0.72$$

$$n \approx \frac{72}{i\%}$$

Compound rate required for doubling amount in specified number of periods:

$$i\% \approx \frac{72}{n}$$

Rule of seven-ten is another rule linking interest and time which states*:*

- Money invested at i=7% annually doubles in about n=10 years
- Money invested at i=10% annually doubles in about n=7 years

Cash Flow Diagrams

To make comparisons between cash flows, we must compare value of money at different points in time. A *cash flow diagram* can help with this.

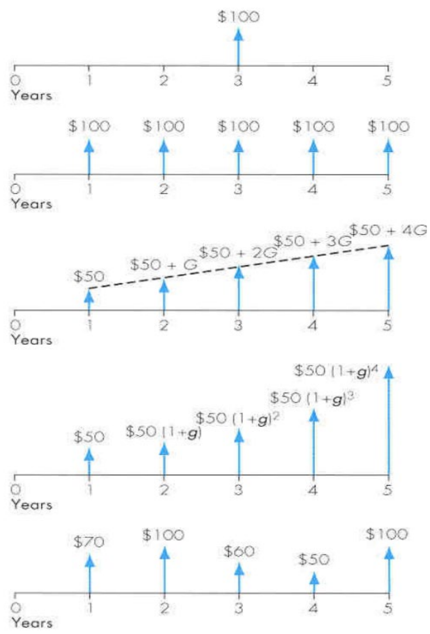
Any downward arrow is cash outflow, and any cash inflow is upward arrow. The time scale shows progression from left to right.

Notation is as follows:

- P=present sum of money
- i=interest rate/period
- N=number of periods
- F=equivalent future sum of money

The formula to calculate the future worth of an investment made today is $F = P(1+i)^N$

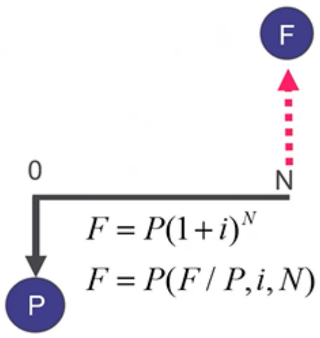
There are different types of cashflows:



- Single cashflow
- Equal (uniform) payment series
- Linear gradient series
- Geometric gradient series
- Irregular payment series

Single Cashflow

Here is an example of a *single payment compound amount factor*. Where i=10%, N=8 years, P=$2000, we can find the future worth of a present value:



We can plug the values into this equation $F = P(1+i)^N$:

$$F = \$2000(1 + 0.10)^8$$

Interest tables can be used to calculate the value in the brackets. The notation ***(F/P,I,N)*** means find *F* given *P* at *i* interest per period for *N* periods.

$$= \$2000(F/P, 10\%, 8)$$

$$= \$4287.18$$

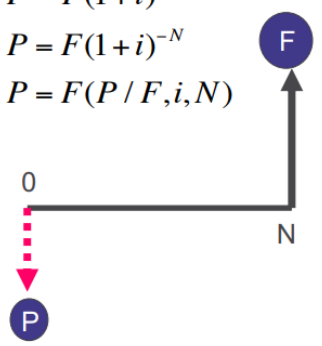
Note that this is what a lookup interest table looks like, and is used to find different values:

TABLE C-13 Discrete Compounding; i = 10%

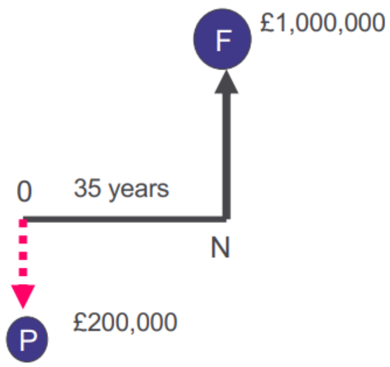
	Single Payment		Uniform Series				Uniform Gradient		
	Compound Amount Factor	Present Worth Factor	Compound Amount Factor	Present Worth Factor	Sinking Fund Factor	Capital Recovery Factor	Gradient Present Worth Factor	Gradient Uniform Series Factor	
N	To Find F Given P F/P	To Find P Given F P/F	To Find F Given A F/A	To Find P Given A P/A	To Find A Given F A/F	To Find A Given P A/P	To Find P Given G P/G	To Find A Given G A/G	N
1	1.1000	0.9091	1.0000	0.9091	1.0000	1.1000	0.000	0.0000	1
2	1.2100	0.8264	2.1000	1.7355	0.4762	0.5762	0.826	0.4762	2
3	1.3310	0.7513	3.3100	2.4869	0.3021	0.4021	2.329	0.9366	3
4	1.4641	0.6830	4.6410	3.1699	0.2155	0.3155	4.378	1.3812	4
5	1.6105	0.6209	6.1051	3.7908	0.1638	0.2638	6.862	1.8101	5
6	1.7716	0.5645	7.7156	4.3553	0.1296	0.2296	9.684	2.2236	6
7	1.9487	0.5132	9.4872	4.8684	0.1054	0.2054	12.763	2.6216	7
8	2.1436	0.4665	11.4359	5.3349	0.0874	0.1874	16.029	3.0045	8
9	2.3579	0.4241	13.5795	5.7590	0.0736	0.1736	19.422	3.3724	9
10	2.5937	0.3855	15.9374	6.1446	0.0627	0.1627	22.891	3.7255	10
11	2.8531	0.3505	18.5312	6.4951	0.0540	0.1540	26.396	4.0641	11
12	3.1384	0.3186	21.3843	6.8137	0.0468	0.1468	29.901	4.3884	12
13	3.4523	0.2897	24.5227	7.1034	0.0408	0.1408	33.377	4.6988	13
14	3.7975	0.2633	27.9750	7.3667	0.0357	0.1357	36.801	4.9955	14
15	4.1772	0.2394	31.7725	7.6061	0.0315	0.1315	40.152	5.2789	15
16	4.5950	0.2176	35.9497	7.8237	0.0278	0.1278	43.416	5.5493	16

With single cashflows, we can also calculate the present value if we know the future value. We make P the subject of the equation. Again, the lookup interest table can be used to find the value in the brackets.

$$F = P(1+i)^N$$
$$P = F(1+i)^{-N}$$
$$P = F(P/F, i, N)$$



We can also calculate an interest rate if we know present and future values, along with the time period between them. For example, take the cashflow diagram below:

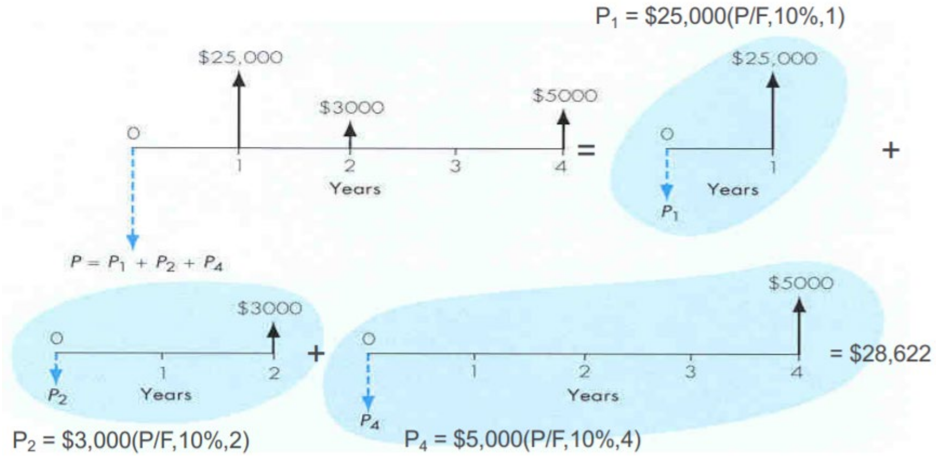


$$£1{,}000{,}000 = £200{,}000(1+i)^{35}$$

We can then solve for i. Note that the interest tables cannot help when solving for i.

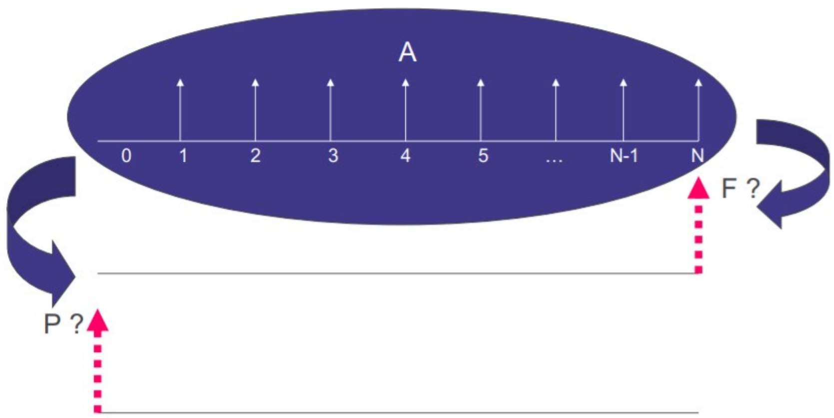
Irregular Payment Series

Cash flow diagrams can be used to deconstruct an uneven payment series. It can be split into single payment cashflow diagrams, and then each one can be solved. For example:



Equal (Uniform) Payment Series

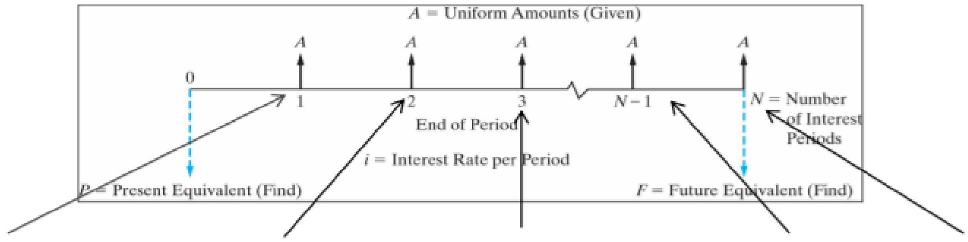
For equally payments (uniform cash flow series), the amount every year is the same but accounts for the principal and the interest that you need to pay back. You can think of this as someone who regularly makes deposits into their pension account, and might want to find the future worth of it.



A uniform cash flow series can also be called an annuity, hence the notation, A.

Given a uniform end-of-period series of payments, we want to find the equivalent single sum amount occurring at the same time as the last payment, F, in the series or at the start, P.

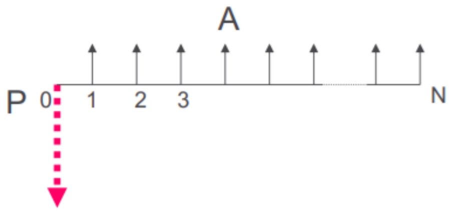
The diagram below shows the derivation of how we can calculate the future worth of a uniform payment series. It is essentially a sum of geometric series:



$$F = A(F/P,i\%,N-1) + A(F/P,i\%,N-2) + A(F/P,i\%,N-3) + \ldots + A(F/P,i\%,1) + A(F/P,i\%,0)$$
$$= A[(1+i)^{N-1} + (1+i)^{N-2} + (1+i)^{N-3} + \ldots + (1+i)^{1} + (1+i)^{0}]$$

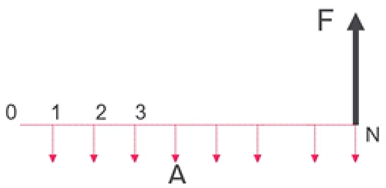
$$\text{sum of geometric series} \Rightarrow S_N = \sum_{k=0}^{N-1} ar^k = a\left[\frac{1-r^N}{1-r}\right] \Rightarrow F = A\left[\frac{1-(1+i)^N}{1-(1+i)}\right] = A\left[\frac{(1+i)^N - 1}{i}\right]$$

To find the present worth of an annuity, we can either use this equation, use use the interest lookup tables, with the form of *(P/A,I,N),* which means find *P* given *A*.



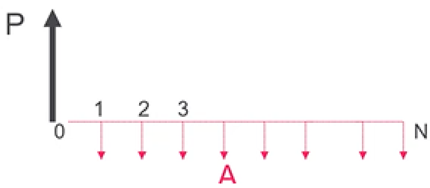
$$F = P(1+i)^N = A\left[\frac{(1+i)^N - 1}{i}\right]$$
$$\Rightarrow P = A\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right] = A(P/A,i,N)$$

This was a method of finding the present value if we know *A*. If we want to find the future value given *A,* this is commonly known as a *sinking fund*. They often used by cities to fund projects, in this you want to have a certain amount, *F*, by a certain amount of time, *N*, and you know you can invest money at a certain interest rate, *i*. We can use the following equation:



$$A = F\left[\frac{i}{(1+i)^N - 1}\right]$$
$$= F(A/F,i,N)$$

Also, we can ask what is *A*, if we need to pay back a specific value *F* over a set time *N* with an interest rate *i*. We can also find a present value, *P,* if we know the annuity.



$$A = P\left[\frac{i(1+i)^N}{(1+i)^N - 1}\right]$$
$$= P(A/P,i,N)$$

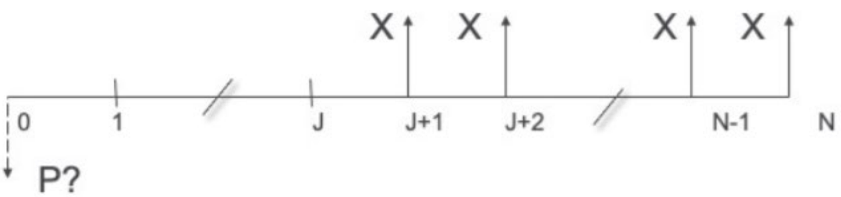
This table summarises the interest factors for single cash flows, and for uniform series:

To Find:	Given:	Factor by which to Multiply "Given"[a]	Factor Name	Factor Functional Symbol[b]
For single cash flows:				
F	P	$(1+i)^N$	Single payment compound amount	$(F/P, i\%, N)$
P	F	$\frac{1}{(1+i)^N}$	Single payment present worth	$(P/F, i\%, N)$
For uniform series (annuities):				
F	A	$\frac{(1+i)^N-1}{i}$	Uniform series compound amount	$(F/A, i\%, N)$
P	A	$\frac{(1+i)^N-1}{i(1+i)^N}$	Uniform series present worth	$(P/A, i\%, N)$
A	F	$\frac{i}{(1+i)^N-1}$	Sinking fund	$(A/F, i\%, N)$
A	P	$\frac{i(1+i)^N}{(1+i)^N-1}$	Capital recovery	$(A/P, i\%, N)$

[a] i equals effective interest rate per interest period; N, number of interest periods; A, uniform series amount (occurs at the end of each interest period); F, future equivalent; P, present equivalent.

Deferred Annuities

For *deferred annuities*, the cash flow is deferred for **J periods** *(J<N)*. So, for example say you took out a mortgage for £200,000, but instead of paying it back in equal amounts from the moment you took it you, you postponed the annual payments by *J* periods.



The present value for the *N-J* uniform annuities of value *X* at time *J* is:

$$X[P/A, i, N-J]$$

This must be further discounted from *J* to the present *P* by multiplying by *[P/F, i, J]*. Therefore:

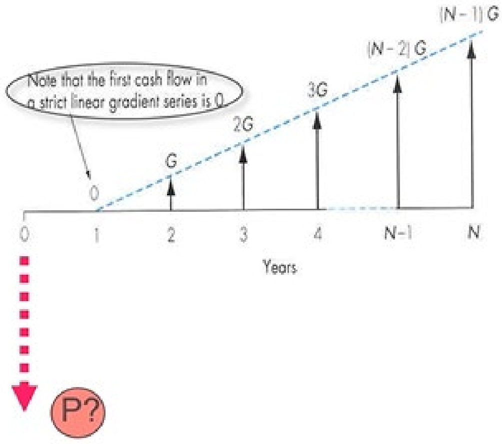
$$P = X[P/A, i, N-J][P/F, i, J]$$

Linear Gradient Payment Series

Linaer gradient series is another type of cashflow. It has the formula:

$$P = G\left\{\frac{1}{i}\left[\frac{(1+i)^N - 1}{i(1+i)^N} - \frac{N}{(1+i)^N}\right]\right\}$$

$$P = G(P/G, i\%, N)$$



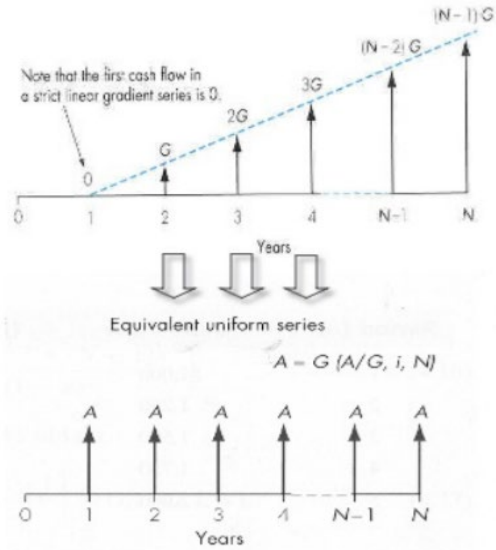
Note that the first positive cash flow always is at year 2. In year 1 it is 0.

We can also convert the gradient series into the equivalent uniform series, finding A:

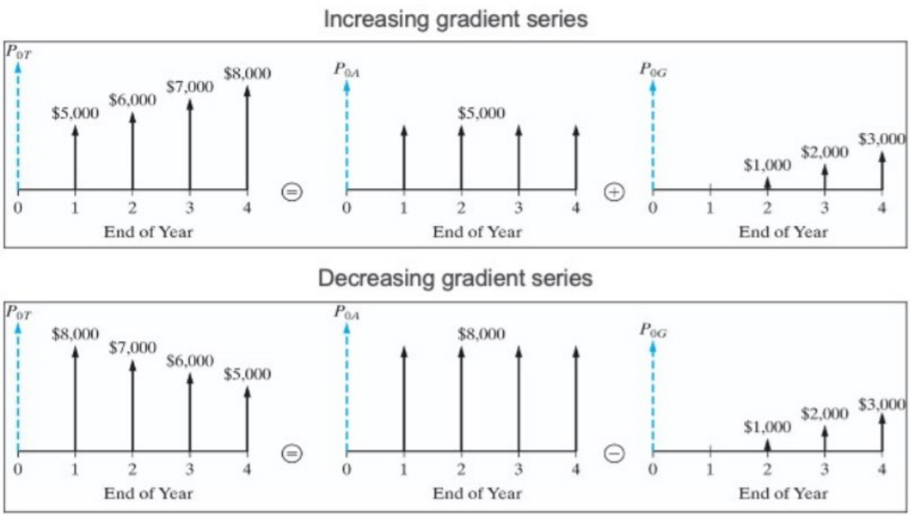
$$A = P(A/P, i\%, N)$$

$$\Rightarrow A = G\left[\frac{1}{i} - \frac{N}{(1+i)^N - 1}\right]$$

$$A = G(A/G, i\%, N)$$

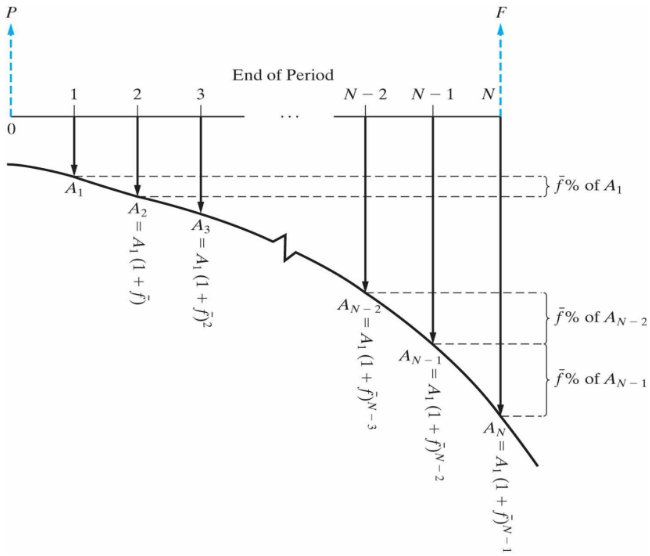


We can also decompose a more complex gradient series as composite series. In the first row, we can see the gradient series can be summed from a uniform payment series, and a linear gradient payment series. A decreasing gradient as seen on the bottom row is a uniform payment series, minus a linear gradient payment series:



Geometric Gradient Series

A *geometric sequence* grows at a compounded rate, so instead of a fixed amount you add, it's a percentage that you multiply every time



It is important not to confuse *F* the future value, with *f* which is the percentage increase in each payment. If the interest rate *i* isn't the same at *f*, then the top formula should be used to calculate the present value, but if *i=f*, then the bottom formula can be used:

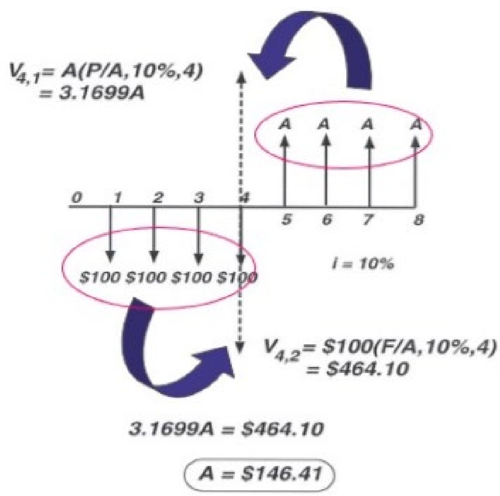
$$P = \begin{cases} \dfrac{A_1[1-(P/F, i\%, N)(F/P, \bar{f}\%, N)]}{i-\bar{f}} & \bar{f} \neq i \\ A_1 N (P/F, i\%, 1) & \bar{f} = i \end{cases}$$

Unconventional Equivalence Calculations

Now that we have looked at some difference types of cashflows, we can look at unconventional equivalence calculations.

An example situation of this is where you make 4 annual deposits of $100 which earns 10% annual interest, and you want to find what equal annual amount, *A,* that you can withdraw over 4 subsequent years.

We can actually do the analysis from any point in time that we want. So, the diagram on the right uses year 4 as the point to equate the two cash flows. But you can also equate them at year 0.



Multiple Compounding in One Period

Interest rates are normally quoted on annual basis as we have seen so far in the cash flow diagrams. However, they may also be compounded several times a year.

Nominal interest (r) is interest compounded more than one interest period per year, but quoted on an annual basis. For example, if you have an annual nominal interest rate of 8% but compounded quarterly, then the 8% is divided into the 4 quarters, giving 2% per quarter.

The effective interest rate (i) is the *actual* interest earned or paid in a year or some other period. So, the actual interest rate in the example given just above is 2% per quarter.

It may seem that over a year, the 2% per quarter is the same as saying 8% per year, but as we know from compounding, it will actually amount to more. If we start with £1000:

$$P = £1000$$

$$£1000(1 + 0.02) = £1020$$

$$£1020(1 + 0.02) = £1040$$

$$£1040(1 + 0.02) = £1061$$

$$£1061(1 + 0.02) = £1082$$

$$\mathrm{i} = \left[\frac{1082 - 1000}{1000}\right] * 100 = 8.2\%$$

We can see from the example that the effective annual interest rate, i, is actually 8.2%, so higher than the nominal interest rate.

The **formula** for the effective annual interest rate, i, is:

$$\mathrm{i} = (1 + \frac{r}{M})^{M} - 1$$

Where M is number of compounding periods per year, r is the nominal interest rate, and r/M is the interest rate per interest period.

From looking at this formula, we can also observe that as M increases, i also increases. So if we take a nominal rate of 18% per year, we can see this effect:

- Compounded semi-annually, i=18.81%
- Compounded quarterly, i=19.25%
- Compounded monthly, i=19.56%
- Compounded daily, i=19.71%

In general, the effective interest rate for a particular payment period is given by:

$$i = \left[1 + \frac{r}{CK}\right]^C - 1$$

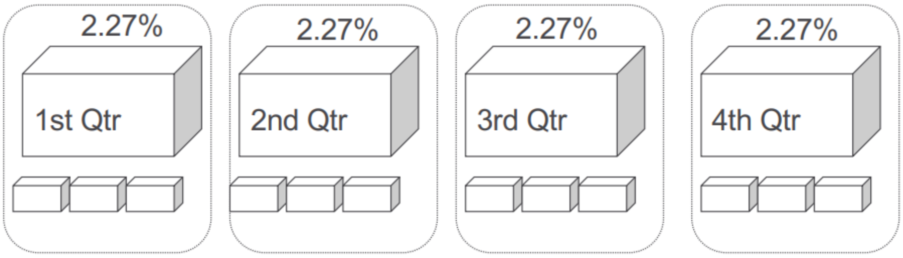
Where r is nominal interest rate, K is number of payment periods per year, and C is number of interest periods per payment period.

For example, if you make *quarterly* deposits in an account that earns 9% annual nominal interest, and its compounded *monthly*:

- Number of interest periods per cash flow periods = 3
- Number of cash flow periods per year = 4

The effective interest rate per quarter is:

$$i = \left[1 + \frac{0.09}{3 * 4}\right]^3 - 1 = 2.27\%$$



Continuous Compounding

We have seen the effect when there is multiple compounding within a specific period, e.g. compounding monthly over a period of a year. It was also observed that as compounding periods, M, increases, interest increases. So, what happens when we there is continuous compounding? We get the formula:

$$i = \lim_{C \to \infty} \left[\left(1 + \frac{r}{CK}\right)^C - 1\right] = e^{r/K} - 1$$

For effective annual interest rate where *K=1* (as payment only occurs once per year), then:

$$i_a = e^r - 1$$

For discrete cashflows, there are 6 formulas for the continuous compounding of these, as opposed to the single annual interest applied earlier in the notes.

$[F/P,\underline{r}\%,N] = e^{rN}$	$[P/A,\underline{r}\%,N] = \frac{e^{rN}-1}{e^{rN}(e^r-1)}$
$[P/F,\underline{r}\%,N] = e^{-rN}$	$[A/F,\underline{r}\%,N] = \frac{e^r-1}{e^{rN}-1}$
$[F/A,\underline{r}\%,N] = \frac{e^{rN}-1}{e^r-1}$	$[A/P,\underline{r}\%,N] = \frac{e^{rN}(e^r-1)}{e^{rN}-1}$

L6. Single Evaluation

The key objective of this section is to learn how to evaluate engineering projects.

Rate of Return is defined as a relative percentage method which measures the yield as a percentage of investment over the life of a project. For example, investing $1650 in Walmart in 1970 was then worth $13,312,000 in 2000.

From this, we can calculate the investments rate of return. This is a single cash flow as we learnt about in the previous section:

Given: P = $1,650
F = $13,312,000
N = 30

Find i: ???

$$F = P(1+i)^N$$

$\$13{,}312{,}000 = \$1{,}650\,(1+i)^{30}$
$(\$13{,}312{,}000 / \$1{,}650)^{1/30} = (1+i)$
$1.3497 = (1+i)$

i = 34.97% → Rate of Return

If you invested the same amount in a savings account at 6% in 1970, your investment would be worth $9477 in the year 2000 (much less!). We can call the 6% alternative investment the opportunity cost (i.e. you gave up investment opportunity at 6% annual return).

Minimum Attractive Rate of Return

The *minimum attractive rate of return* (MARR) is the return we are requiring on that investment considering we have other alternatives. So, the return in Walmart must be at least 6%. This is also known as **hurdle rate, discount rate, opportunity cost of capital.**

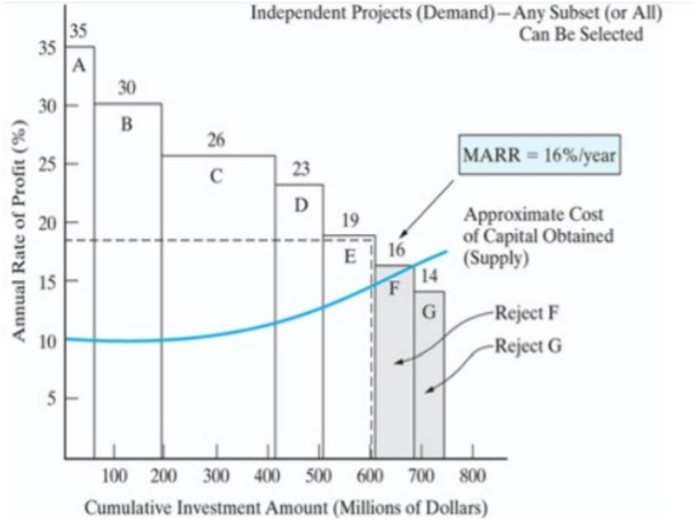
Think of it on a personal level. Say you maybe have £500 in saving now in 2022. You could put them in a savings account that guarantees say 3% interest. Or you could choose to invest them in stocks. If you choose to invest them in stocks (which are also higher risk), you would want the value of those stocks to increase by 3% at a minimum, otherwise you would have been better off putting that £500 in a savings account.

MARR in businesses can be decided by the amount and cost of money a company obtains (e.g. weighted average cost of capital). For example, if a company borrows money at 8%, it must invest the money in projects giving at least 8% returns. It can also be decided by return rate of the next best investment e.g. two projects with similar risk, if one is at *5 the other must be higher to invest in it.

As we know MARR is the return provided by the next best available investment opportunity.

As an example, if we had a budget of 600 million, we would invest in A, B, C, D and E as they give the highest rate of profit, but F is the next highest at 16 so this is MARR.

The solid line shows that the cost of capital typically increases as more money is borrowed through debt and/or equity issuance.



There is different MARR for different **risks**. Higher risk projects should be compensated with higher annual rate of return.

A firm may set different MARRs according to the level of risk involved, e.g.:

- High risk MARR 40% for new products, new business, acquisitions, joint ventures
- Moderate risk MARR 25% for capacity increase to meet forecasted sales
- Low risk MARR 15% for cost improvements, make versus buy, capital increase to meet existing order

Weighted Average Cost of Capital

The WACC is one was to calculate MARR, and the basic idea is that it's an average expected return based on *how much money cost you when you borrow it*.

As mentioned, the two common ways a firm raises money is through:

- Equity (through selling company shares)
- Bonds (like a loan where the company pays interest plus the principal)

The formula for WACC is:

$$WACC = R\ on\ equity * Equity\ \% + R\ on\ bonds * Bonds\ \%$$

Now, the return on equity is difficult to estimate:

- Estimate future growth and earnings, based on track record and prospects
- Examine historical returns for similar companies in similar situations

As an example, a company sells £10 million in shares with estimated dividend paid as a return at 25%, and issuing £5 million in debt (bonds), and will pay 10% a year. Their total money railed = debt + equity = £15 million.

$$WACC \;=\; 25\%(2/3) + 10\%(1/3) \;=\; 20\%$$

From this, we can see that the companies WACC is 20%, their 'cost of borrowing', and therefore this can be used as a MARR, so any investment made by the company using this money must generate at least 20% return.

Present Worth Analysis

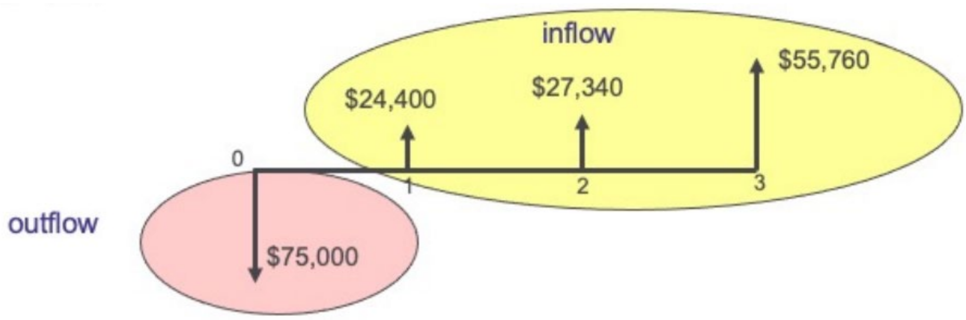
Present worth (PW) analysis is based on a concept of equivalent worth of all cash flows relative to the present as a base. Think of *P,* in the previous section.

All cash inflows and outflows are discounted to present, generally at an i equal to the MARR. PW is a measure of how much money can be afforded for investment

PW is positive if the sum of discounted cash inflows outweighs the sum of discounted cash outflows over the project lifecycle. The discount rate i is assumed constant over the project. The higher the discount rate and further into a cash flow occurs, the lower its present worth.

The decision rule is to accept the project if net surplus is >=0.

An example will help to demonstrate this. Say a company is considering buying a new machine. The initial investment is $75,000 and projected cash flows are $24,000, $27340 and $55760 over years 1, 2, 3 respectively. The MARR = 15%. We can use the cash flow series in the previous section to solve this and find the present worth.



$$\begin{aligned}
PW(15\%)_{\text{inflow}} &= \$24,400(P/F,15\%,1)+\$27,340(P/F,15\%,2) \\
&\quad +\$55,760(P/F,15\%,3) \\
&= \$78,553 \\
PW(15\%)_{\text{outflow}} &= \$75,000 \\
PW(15\%) &= \$78,553-\$75,000 \\
&= \$3,553>0,\ \text{Accept}
\end{aligned}$$

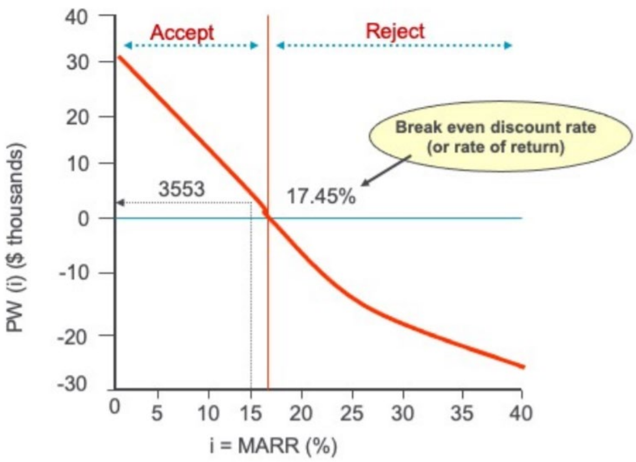
We can see that the PW is $3553. As this is greater than 0, we can accept the investment.

Carrying on with this example, we could repeat the calculation with different values of *i.* Here we can see the effect of present worth amounts at varying discount rates:

i (%)	PW(i)
0	32,500
2	27,743
4	23,309
6	19,169
8	15,296
10	11,670
12	8,270
14	5,077
16	2,076
17.45*	0
18	-751
20	-3,412

Project remains profitable so long as i < 17.45%, as positive PW indicates

We can see that when i = 17.45%, the present worth is 0. This can be thought of as a company having to pay an interest of 17.45% to borrow the capital used in the investment. If the cost of borrowing is any higher, then it would not be worth investing in the machinery as the returns from it aren't great enough. It is visualised below:



There are two other types of analysis:

1. Future worth is based on the equivalent worth of all cash inflows and outflows at the end of the planning horizon, compounded at an interest rate that is generally MARR. Similarly to present worth, if FW>=0, the project/investment is economically justified.

2. Annual worth of a project is an equal annual series of dollar amounts, over a stated period N, equivalent to the cash inflows and outflows at interest rate that is generally MARR. If AW>0, the project is economically attractive. If you are comparing a project that has 4 years of life and an 8-year project annual worth is better to use AW to compare.

It is important to note that PW, AW and FW are equivalent in that they always give the same recommendation since each is related to the other two via a constant multiplier.

Annual Worth Applications

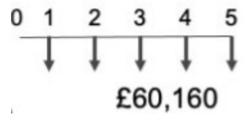
Capital recovery is the cost of owning an equipment associated with two transactions – initial cost (I), and salvage value (S). Say a company owns a 3D printer, the initial money they spend on buying it is I, and the money that they can earn from selling it is S.

Annual capital costs are calculated as:

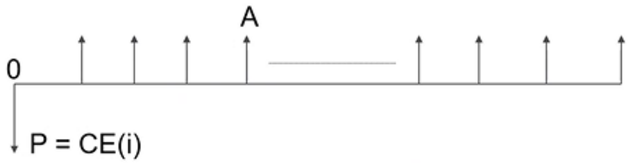
$$CR(i\%) = I\left(\frac{A}{P}, i\%, N\right) - S\left(\frac{A}{F}, i\%, N\right)$$

For example, £200,000 put into a project, N=5, equipment can be sold for £50,000 and i=20% (MARR):

CR(i) = I (A/P, i, N) – S (A/F, i, N)
= 200,000 (A/P, 20%, 5) – 50,000 (A/F, 20%, 5)
= 200,000 (0.3344) – 50,000 (0.1344)
= £60,160

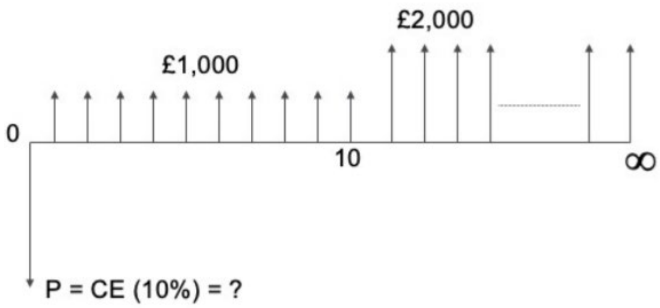


Some projects may have a very long life e.g. tunnels or bridges. In this case, a special version of PW method is used, **Capitalized equivalent worth.** Time is assumed *infinite*. The principle is that PW for a project with an annual receipt of A over infinite service life:

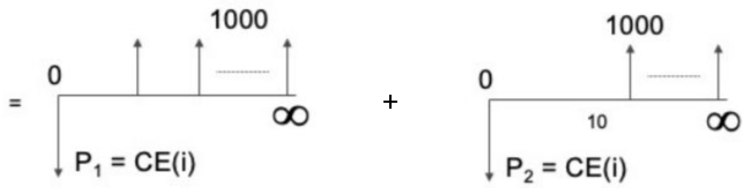


$$P = CE(i) = A\left(\frac{P}{A}, i\,, \infty\right) = \frac{A}{i}$$

As an example, say a project will generate £1000 for 10 years and £2000 until infinity.



This can be split into two sections. £1000 from 0 to infinity. And another 1000 from N=10 to infinity.



CE(10%) = 1,000 / 0.1 + [1,000/0.1] [P/F,10%,10]
= 10,000 + 10,000 (0.3855)
= £13,855

It is worth noting that a project with a service life of say 50 years wouldn't be that much more net worth if it was until infinity as it gets discounted by the MARR so heavily that it doesn't add much to the present/future/annual net worth from the years; 50 to infinity.

Internal Rate of Return

IRR is the break-even discount rate, *i**, which equates the present worth of a projects cash *outflows* to the present worth of its cash *inflows*. The relationship can be seen mathematically:

$$PW(i^*)_{cash\ inflows} \qquad = \qquad PW(i^*)_{cash\ outflows}$$

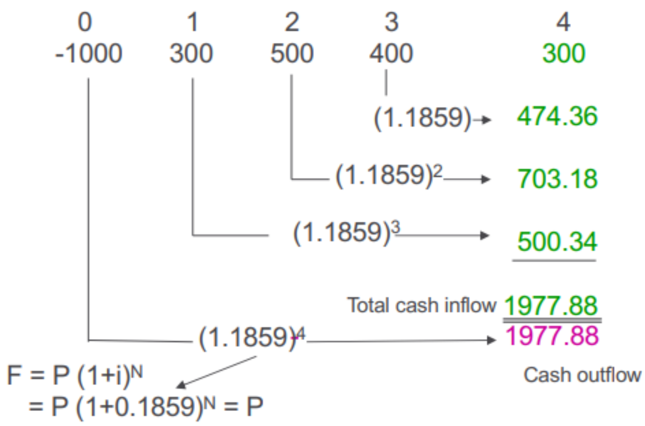
The decision rule is to accept project if IRR>=MARR. When you use the IRR you are calculating the rate at which the present worth of a project is 0. As we saw in the example earlier when *i*=17.45% the present worth was 0 and therefore 17.45% was the IRR.

There are however some issues with IRR (*i**). They include:

- Computation of IRR may be difficult
- Multiple IRRs may be calculated for the same problem
- IRR is positive for a single alternative only if both receipts are present in the cash flow pattern and the sum of receipts exceeds sum of cash outflows.
- The IRR methods must be carefully applied and interpreted in the analysis of two or more alternatives, where only one is acceptable.
- The IRR method assumes recovered funds, if not consumed each time, are reinvested at i*%, rather than at MARR. This might be unrealistic if IRR>MARR.

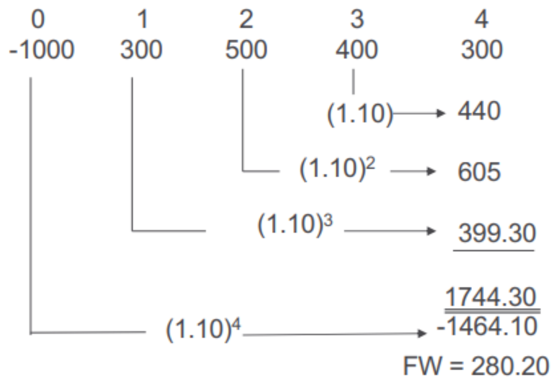
The last issue is a key difference between the IRR and PW methods. In the IRR method, since we assume that the present worth can inflows equal the present cash outflows, then the future cash inflows must also equal future cash outflows.

Say an IRR=18.59%, the future worth's from each year would look like this:



If we look at year 3 for example with the inflow if 400, this is multiplied by the IRR of 18.59% to find its future worth in year 4. So we can see how the IRR method assumes that funds are reinvested at *i**, rather than at MARR which may be unrealistic if IRR>MARR.

In the PW method, we can see how this is different. If the MARR=10%, then:



We can see that this method assumes that funds, if not consumed, are reinvested at the MARR %. This is more realistic.

There is also a **criterion for the uniqueness** of the IRR. In computing the i^*, we are solving the following equation (which can also be observed from the above diagrams):

$$F_0 + F_1 x + F_2 x^2 + \ldots + F_N x^N = 0, \qquad where\ x = 1/(1+i^*)$$

We are interested in the number of positive real roots of an n-degree polynomial which gives us the future worth.

Descartes rule states the number of real *positive* roots of an n-degree polynomial with real coefficients is never greater than the number of sign-changes in the sequence of the coefficient. Hence, we just need to examine the number of sign changes in F_k, $k=1 \ldots N$

This rule can be seen here, where the unique IRR only has one sign change in the net cash flow series, whereas when there is more than one sign change in the cash flow series, there is the possibility of multiple IRR:

A Unique IRR	Possibility of multiple IRRs
• Initial cash flows are negative, and only one sign change occurs in the **net** cash flows series	• Initial cash flows are negative, but more than one sign changes in the remaining cash flow series.
• Example: -£100, £250, £300 (-, +, +)	• Example: -£100, £300, -£120 (-, +, -)

Here in this example the number sign changes is 3, and therefore by using Descartes rule, then i*<=3:

n	Net Cash flow	Sign Change
0	-£100	
1	-£20	
2	£50	1
3	0	
4	£60	
5	-£30	1
6	£100	1

The **Norstrom criterion** is a more discriminating condition. It applies with the *sum* of all the cash flows rather than each individual annual cash flow, and states if the series S starts negatively and changes sign only once, there exists a unique positive *i**.

Period (n)	Cash Flow (A_n)	Sum S_n
0	A_0	$S_0 = A_0$
1	A_1	$S_1 = S_0 + A_1$
2	A_2	$S_2 = S_1 + A_2$
$\vdots$	$\vdots$	$\vdots$
N	A_N	$S_N = S_{N-1} + A_N$

In this example, even though there are 3 sign changes in the cash flow A_n, we can see that in the *sum* column, there is 1 sign change, so there is a unique *i**.

n	A_n	S_n	Sign change
0	-£100	-£100	
1	-£20	-£120	
2	£50	-£70	
3	0	-£70	
4	£60	-£10	
5	-£30	-£40	
6	£100	£60	1

Calculating the IRR

There are a number of different methods to calculate the IRR:

- Direct solution
- Trial and error
- Graphical solution
- Computer software (excel – Goal seek, IRR, MIRR)

1. Direct Solution example. Note that as N increases, the equation get harder to solve!

n	Cash flow
0	-£2,000
1	1,300
2	1,500

$PW_{outflow} = PW_{inflow}$

2,000 = 1,300 (P/F, i%, 1) + 1,500(P/F, i%, 2)

$2{,}000 = 1{,}300 / (1+i)^1 + 1{,}500 / (1+i)^2$

Since $P = F / (1+i)^N$

$$PW(i) = -\$2{,}000 + \frac{\$1{,}300}{(1+i)} + \frac{\$1{,}500}{(1+i)^2} = 0$$

Let $x = \frac{1}{1+i}$, then

$$PW(i) = -2{,}000 + 1{,}300x + 1{,}500x^2$$

Solve for x:

$x = 0.8$ or -1.667

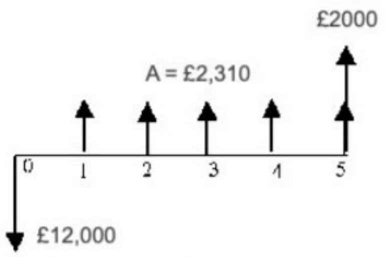
Solving for i yields

$$0.8 = \frac{1}{1+i} \rightarrow i = 25\%, \quad -1.667 = \frac{1}{1+i} \rightarrow i = -160\%$$

Since $-100\% < i < \infty$, the project's $i^* = 25\%$

Also note the minimum bound that *i* can be is -100%, as you cannot lose more than 100% of your money. However, you can make more than 100% of your money, hence the upper bound being infinity.

2. Trial and Error method example. This method is fairly self-explanatory.



If MARR = 10%, should this project be accepted?

Find i* such that PW(i*%) = 0

-12,000 + 2,310 [P/A, i*, 5] + 2,000 [P/F, i*, 5] = 0

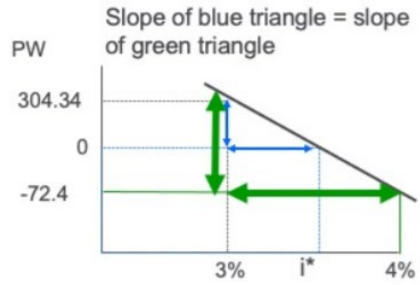
Try i* = 3%: -12,000 + 2,310 (4.5797) + 2,000 (0.8626) = 304.34 >0
Try i* = 5%: -12,000 + 2,310 (4.3295) + 2,000 (0.7835) =-431.86 <0
Try i* = 4%: -12,000 + 2,310 (4.4518) + 2,000 (0.8219) =-72.44 <0

Hence 3% < i* < 4% < MARR: project should be REJECTED

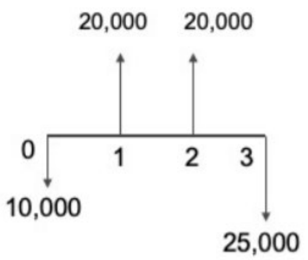
Trial and error can also be done with *linear interpolation***.** The 'goal seek' function on Excel does this which is fast and useful when there are too many N for the direct solution method.

$$\frac{i^*-3}{304.34-0}=\frac{4\%-3\%}{304.34-(-72.4)}$$

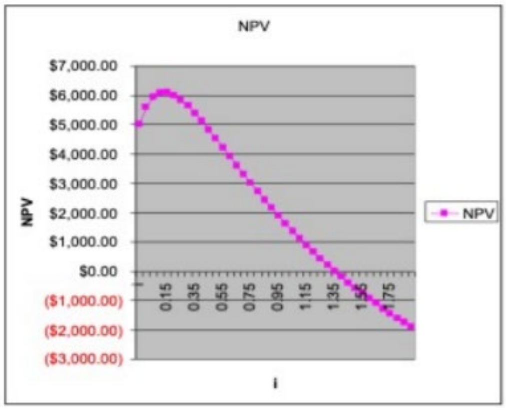
$$i^*=\left[\frac{4\%-3\%}{304.34+72.4}(304.34)\right]+3\%=3.8\%$$



3. Graphical solution is to create a PW plot in say Excel, and then identifying the point at which the curve crosses the horizontal axis to approximate solution *i*.*



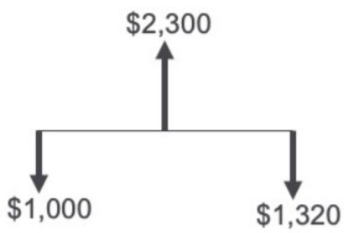
PW(i) = init. invest + NPV (rate, values)



Note that the graph shows the NPV against the interest rate, not time. So for example when i=0, then NPV= £5000 as it is simply -10,000+20,000+20,000-25,000=5000.

Multiple IRR

So, what do we do when we are presented with multiple IRR's? When there is multiple IRR solutions to a problem, it's hard to know the correct solution:



$$PW(i)=-\$1{,}000+\frac{\$2{,}300}{1+i}+\frac{-\$1{,}320}{(1+i)^2}$$
$$=0$$

Let $x=\dfrac{1}{1+i}$. Then,

$$PW(i)=-\$1{,}000+\frac{\$2{,}300}{(1+i)}-\frac{\$1{,}320}{(1+i)^2}$$
$$=-\$1{,}000+\$2{,}300x-\$1{,}320x^2$$
$$=0$$

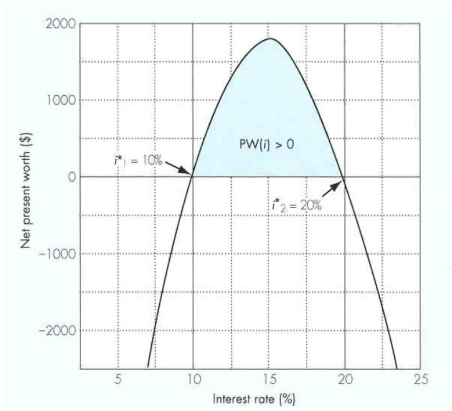
Solving for x yields,

$x=10/11$ or $x=10/12$

Solving for i yields

$i=10\%$ or 20%

This specific example can be plotted, showing the two IRR's of 10% and 20%:



If you encounter multiple rates of return, abandon IRR analysis and use PW (or NPV) criterion. If the MARR = 15% in this specific example:

$$
\begin{aligned}
PW(15\%) = & -£1{,}000 \\
& + £2{,}300\ (P/F, 15\%, 1) \\
& - £1{,}320\ (P/F, 15\%, 2) \\
& = £1.89 > 0
\end{aligned}
$$

So, accept the investment. Note that in this example 10%<MARR<20% will give positive PW.

Payback Period

So far, when looking at PW, FW and AW and rate or return methods, these are based on profitability. *Payback* is based on *project liquidity*.

The **simple payback period** has that principle of 'how fast can I recover my initial investment'. The downside is that this method doesn't account for discounting, so when done in industry it is not over a long time period.

It is based on undiscounted cumulative cash flow and states that if the payback period is less than or equal to some specified payback period, the project should be considered.

For example, a new equipment is expected to generate the cash flow pattern shown. What is the simple payback?

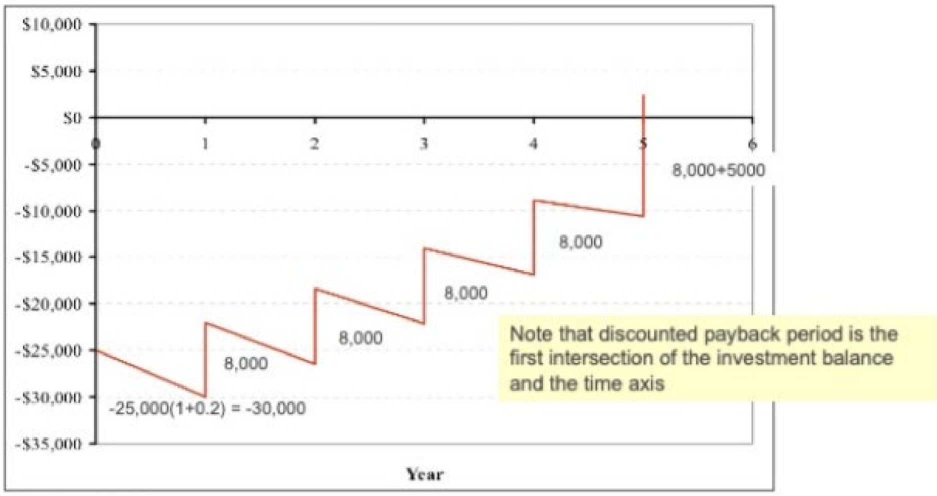
N	Cash Flow	Cum. Flow
0	-£105,000+£20,000	-£85,000
1	£35,000	-£50,000
2	£45,000	-£5,000
3	£50,000	£45,000
4	£50,000	£95,000
5	£45,000	£140,000
6	£35,000	£175,000

Payback period occurs somewhere between N=2 and N=3. The exact time is 2 years +(5000/50000)365=2 years 37 days.

To be more correct, we can use **discounted payback period** as this considers the time value of money. In this example we add a cost of fund column, and base the cumulative cash flow on this:

Period	Cash Flow	Cost of Funds (15%)*	Cumulative Cash Flow
0	-£85,000	0	-£85,000
1	15,000	-£85,000(0.15)= -£12,750	-82,750
2	25,000	-£82,750(0.15)= -12,413	-70,163
3	35,000	-£70,163(0.15)= -10,524	-45,687
4	45,000	-£45,687(0.15)=-6,853	-7,540
5	45,000	-£7,540(0.15)= -1,131	36,329
6	35,000	£36,329(0.15)= 5,449	76,778

And to finish this section, we can use **investment balance diagrams** which describe how much money is tied up in the project at any time, and how fund recovery behaves overestimated life:



L7. Select Alternatives

There is often *more than one option*/alternative/design for *investing* money. It's the design engineers' task to identify and develop good alternatives. These design alternatives are likely to:

- Require different amount of capital investment
- Target different customer types, markets
- Exhibit different patterns of cash flows
- Have different useful lives

There are three groups of major investment projects:

- *Mutually exclusive*: at most one project from a group can be chosen
- *Independent*: choice of the project independent of choice of any other projects in the group, so all or none of the projects may be selected or a number in-between
- *Contingent*: choice of project is conditional on choice of one or more other projects

Mutually Exclusive Projects

Two projects are **mutually exclusive** if selecting one precludes the selection of other alternatives. There are two types of decisions:

- Investment: alternatives with initial capital investments that produce positive cash flows from increased revenue, savings or both.
- Cost alternatives: alternatives with negative cash flows except for a possible cash flow element from disposal of assets at the end of the project's useful life.

Principles of mutually exclusive projects include investing as much capital as possible at rate of return >=MARR.

Another principle is that the alternative requiring *minimum capital investment* and producing *satisfactory results* should be chosen (unless incremental capital associated with an alternative having more investment can be justified with respect to incremental savings).

When selecting mutually exclusive alternatives there are two rules:

- When revenues and other economics benefits are present, select the alternative with greatest positive equivalent worth (PW, FW, AW) at *i*=MARR that satisfies project requirements (i.e. Investment alternatives)
- When revenues and economic benefits are not present, select alternatives that minimises cost (least negative PW, AW, FW) (i.e. Cost alternatives)

There are also two different types of **time horizons**:

- *Study life*: The selected time over which alternatives are compared
- *Useful life*: The time period during which an asset or an alternative is kept in productive use in a trade or business

When comparing mutually exclusive projects, they must be compared over equal time span, so if required service period is given, analysis period should be the same as required service period.

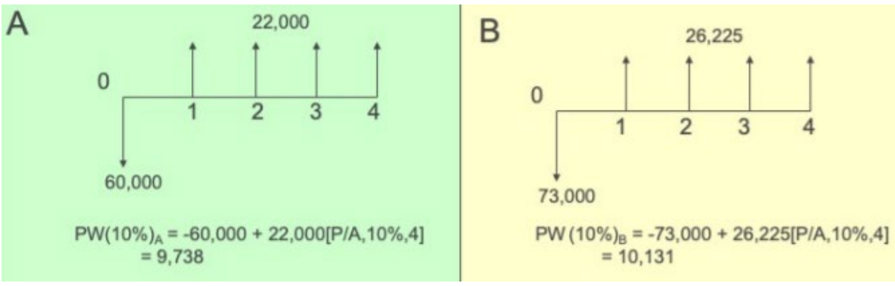
There are two cases for time horizon:

- Case 1: useful lives are the *same* for all alternatives and equal to the study period
- Case 2: useful lives are *different* among alternatives and at least one does not match the study period.

Mutually Exclusive Alternatives: Study Life = Useful Life

We can see an example of Case 1 (study life = useful life) for *investment alternatives*. Whether to choose alternative A or B, with both having N=4 and the MARR=10. Alternative A requires a smaller capital and should be chosen unless the incremental investment of 13,000 for choosing B at the MARR is worthwhile.

	Alternatives	
	A	B
Capital Investment	-60,000	-73,000
Annual revenue – expenses	22,000	26,225



Both A and B are acceptable as their PW>0 but using the investment alternative rule of choosing the alternative with the highest PW, B should be chosen.

Another example when study life = useful life is for *cost alternative***.** This is where the least negative value is chosen to minimise cost. If the MARR=10%:

Year	Alternatives		Diff.
	I	II	II-I
0	-380,000	-415,000	-35,000
1	-38,100	-27,400	10,700
2	-39,100	-27,400	11,700
3	-40,100	-27,400	12,700
Salvage	0	26,000	26,000

I	II
$PW(10\%)_I = -380{,}000$ $- 38{,}100\,[P/F,10\%,1]$ $- 39{,}100\,[P/F,10\%,2]$ $- 40{,}100\,[P/F,10\%,3]$ $= -477{,}077$	$PW(10\%)_{II} = -415{,}000$ $- 27{,}400\,[P/A,10\%,3]$ $+ 26{,}000\,[P/F,10\%,3]$ $= -463{,}602$

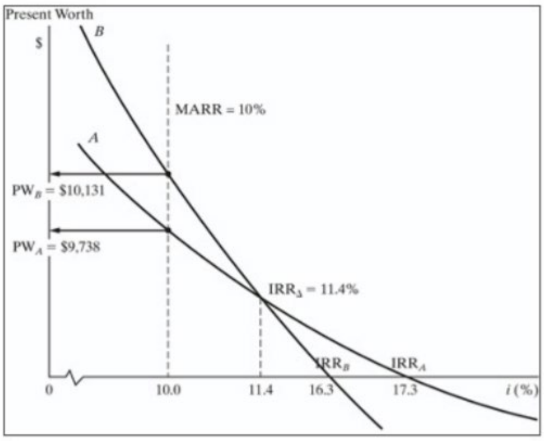
In this situation, alternative II is preferred, as it has the least negative value.

Selection of Mutually Exclusive Alternatives Using IRR

Project selection can also be done using IRR. Maximising IRR is wrong (for mutually exclusive projects)! With the earlier example on the *investment alternative*:

	Alternatives		Diff
	A	B	B-A
Capital Investment	-60,000	-73,000	-13,000
Annual revenue – expenses	22,000	26,225	4,225
PW(10%)	9,738	10,131	393
IRR	17.3%	16.3%	11.4%

By maximising PW, alternative B should be chosen, but B has a lower IRR compared to A. Why is this?



Maximising IRR does not necessarily lead to the correct choice, because of the assumptions made when using this method, such as reinvesting funds at the IRR rate rather that MARR, as discussed in a previous section.

The graph shows the two alternatives A and B. B has a higher PW, but A has a higher IRR. This happens because alternative B has higher upfront investment, but also higher positive future cashflows. Alternative A has less upfront investment but also lower positive future cashflows. So, at low discount rates, *i,* alternative B is favoured, but at high discount rates, B's future cashflows get discounted more, meaning that A becomes more favourable.

In essence, the best alternative always depends on the discount rate, and IRR is just the discount rate at which PW=0, which is why it may not be a good metric for maximisation.

Therefore, if IRR is used, incremental analysis must be used.

Now let's look at how to use IRR incremental analysis. Looking at this example, if we invest an additional £4000 into project A2, you will make an additional £5000, which gives us an incremental IRR of 25%.

n	Project A1	Project A2	Incremental Investment (A2 – A1)
0	-£1,000	-£5,000	-£4,000
1	£2,000	£7,000	£5,000
IRR	100%	40%	25%
PW(10%)	£818	£1,364	£546

25% is higher than the MARR of 10%, and therefore, the incremental investment of A2 is justified.

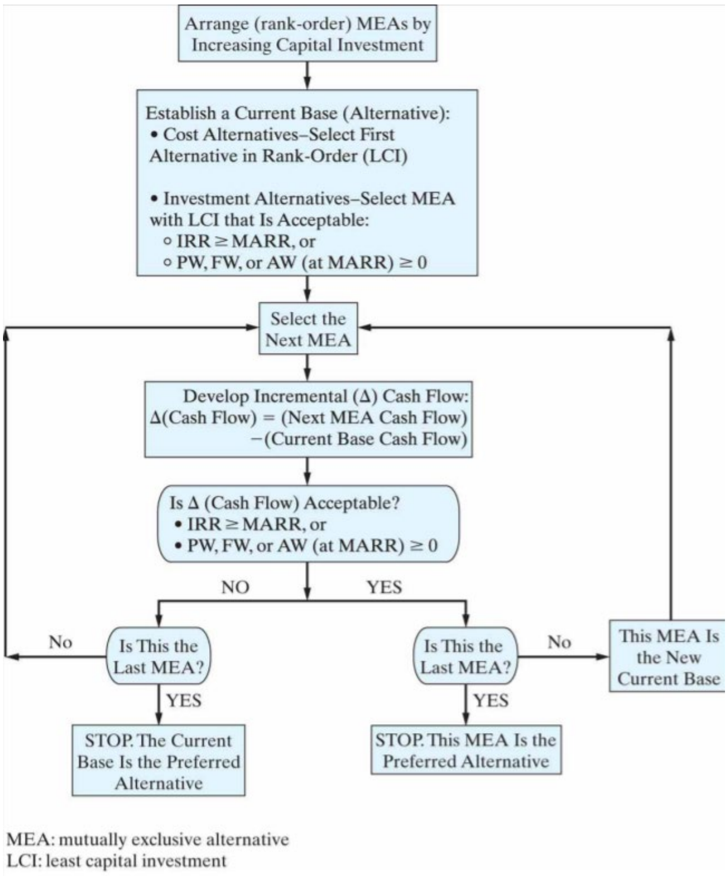
Incremental investment arranges the alternatives in increasing amounts of capital investment. There is a systematical way to do this.

1. Establish a base alternative:

- Cost alternative, the first alternative is the base
- Investment alternative, go down the sorted list and stop at the first alternative which is acceptable i.e. IRR>=MARR; PW, FW, AR at MARR>=0

2. From the base alternative, pick the next acceptable alternative down the sorted list and evaluate the difference between this alternative and the base alternative.

3. If this increment is desirable, the new alternative becomes the base and repeat the process until the list is exhausted.



Let's try an example using this procedure. With a MARR=10%, and N=10 years there are alternatives A, B, C, D, E and F.

	A	B	C	D	E	F
Capital	-900	-1,500	-2,500	-4,000	-5,000	-7,000
Annual income	150	276	400	925	1125	1425
IRR	10.6	13.0	9.6	19.1	18.3	15.6

Step 1: Arrange then according to upfront capital required if not done already. Examine the IRR for each project to eliminate any project failing to meet MARR. So, get rid of C in this case (if the IRR is not given, find them).

	A	B-A	D-B	E-D	F-E
Capital increment	-900	-600	-2,500	-1,000	-2,000
Annual income increment	150	126	649	200	300
IRR (%)	10.6	16.4	22.6	15.1	8.1

Step 2: Compare A and B in pairs. Find IRR_{B-A} = 16.4 >10, so B is better than A.

Step 3: Compare B and D in pairs. Find IRR_{D-B} = 22.6 >10, so D is better than B.

Step 4: Compare E and E in pairs. Find IRR_{E-D} = 15.1 >10, so E is better than D.

Step 5: Compare E and F in pairs. Find IRR_{F-E} = 8.1 >10, so F is worse than E.

Therefore, we can say that alternative E is the best choice!

There is also **Incremental analysis for cost-only** projects. Here is an example:

	Alternatives			
	D_1	D_2	D_3	D_4
Capital	– 100,000	−140,600	– 148,200	– 122,000
Annual expenses	– 29,000	– 16,900	– 14,800	– 22,100
Useful life (years)	5	5	5	5
Market value at end of yr 5	10,000	14,000	25,600	14,000

Order by lowest capital first, and then calculate IRR differences between alternatives:

	Difference		
	D_4–D_1	D_2–D_4	D_3–D_4
ΔCapital	– 22,000	– 18,600	– 26,200
ΔAnnual saving	6,900	5,200	7,300
ΔMarket value	4,000	0	11,600
IRR_Δ	20.5%	12.3%	20.4%
Justified?	yes	no	yes
Choose	D_4	D_4	D_3

As the MARR=20%, we can see that D3 is the best choice.

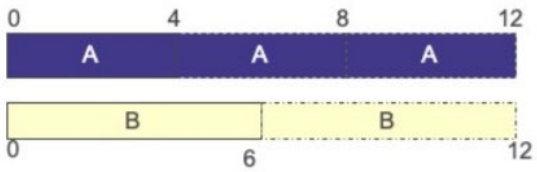
Mutually Exclusive Alternatives: Study Life ≠ Useful Life

Now, as mentioned in the beginning of the section, case 2 is that alternatives useful lives are *different* and at least one does not match the study period.

For this case we use the **repeatability assumption**:

- Study period which the alternatives are being considered is either indefinitely long, or equal to common multiple of the lives of alternatives.
- Economic consequences estimated to happen in an alternatives initial useful lifespan will also happen in all succeeding lifespans

For example, A and B have useful lives of 4 and 6 years respectively. The lowest common multiple of their lives is 12 years:

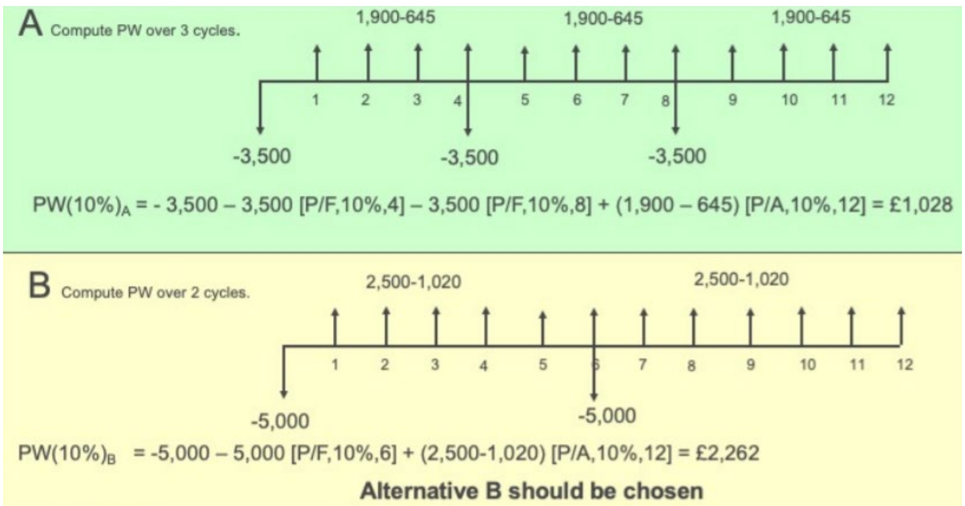


Under the repeatability assumption, the criterion is then to select the alternative with:

- Maximum present worth over 12 years, **or**
- Maximum annual worth, *A*

Let's use the repeatability assumption in this example when MARR = 10%:

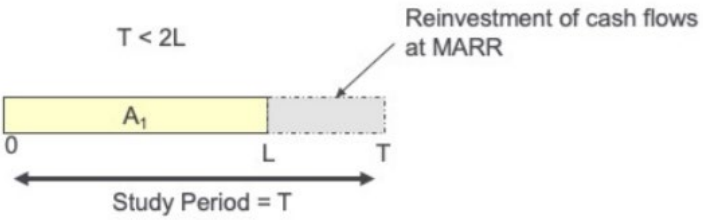
	Alternative	
	A	B
Capital Investment	-3,500	-5,000
Annual Revenue	1,900	2,500
Annual Expenses	-645	-1,020
Useful life (years)	4	6
Market value at end of useful life	0	0



Another assumption that can be used when lives do not match the study period is the **Coterminated assumption**:

- Finite and identical study period is used for all alternatives
- Planning horizon, combined with appropriate adjustments to estimated cash flows, puts the alternatives on a common and comparable basis
- Used when repeatability assumption is not applicable
- Approach most frequently used in engineering practice

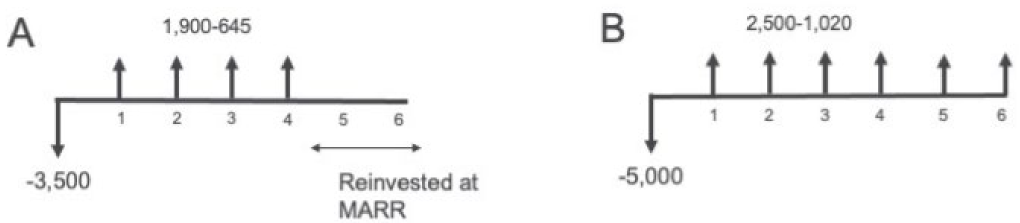
For *investment alternatives*, assume cash flows will be reinvested in other opportunities available at MARR until the end of study period and then compute FW of each project at end of study period.



Let's use the coterminated assumption in this example when MARR = 10%:

	Alternative	
	A	B
Capital Investment	-3,500	-5,000
Annual Revenue	1,900	2,500
Annual Expenses	-645	-1,020
Useful life (years)	4	6
Market value at end of useful life	0	0

Under coterminated assumption with study period = 6 years



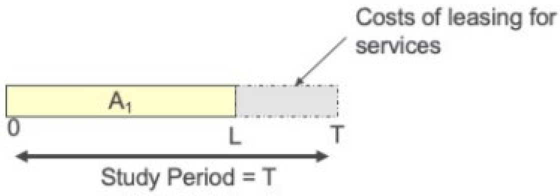
Using FW method

$FW(10\%)_A$ at end of 6 years
= {-3.500 [F/P, 10%, 4] + (1,900 - 645) [F/A,10%,4] } [F/P, 10%, 2]
= £847

$FW(10\%)_B$ at end of 6 years
= -5,000 [F/P, 10%, 6] + (2,500 – 1,020)[F/A, 10%, 6]
= £2,561

B should be chosen!

The above works for *investment alternatives*, but for *cost alternatives*, you cannot reinvest cash flows at MARR. Assume instead of contracting for service or leasing needed equipment for the remaining years of service to end of study period.



Let's use the cotermination assumption in a cost alternative example. Consider a problem with 2 alternatives for the purchase of 4 forklift trucks. Data for 4 trucks are as follows. Study period = 8 years, MARR = 15%:

	Alternatives	
	Stackhigh	S-2000
Capital Investment	-184,000	-242,000
Annual Expenses	-30,000	-26,700
Useful life (years)	5	7
Market value at end of useful life	13,000	5,700

Future leasing for 4 forklift trucks will cost 104000 per year based on 3 year lease, or 134000 per year based on 1 year lease.

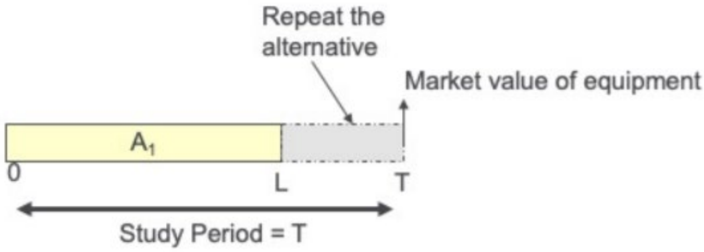
If leasing is used at end of useful life to provide full 8 years of comparable service, which model should be chosen using PW of incremental cash flows?

EOY (Study period)	Alternative cash flows		
	Stackhigh	S-2000	S-2000 – Stackhigh
0	-184,000	-242,000	-58,000
1	-30,000	-26,700	3,300
2	-30,000	-26,700	3,300
3	-30,000	-26,700	3,300
4	-30,000	-26,700	3,300
5	-17,000	-26,700	-9,700
6	-104,000	-26,700	77,300
7	-104,000	-21,000	83,000
8	-104,000	-134,000	-30,000

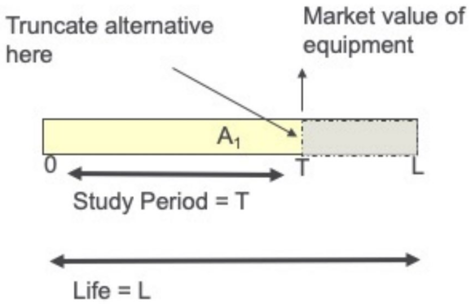
$PW(15\%)_D = 1{,}413 > 0$

We can see that S-2000 should be chosen as the increment of additional capital investment required in S-2000 v Stackhigh is economically justified.

For *cost alternatives*, if the useful life is *less* than the study period, another assumption is to repeat part of useful life of alternative, then use estimated market value to truncate it at end of study period.



If the useful life is *longer* than the study period, the alternative can be truncated at the end of the study period, and an estimated market value can be used assuming the disposable assets will be sold at the end of the study period at that value.



This is what is done in real estate development when salvage value is assumed as if could lease until infinity, which is used to estimate market value in a particular year.

Lets look at an example where the study period is 2 years and MARR=15%:

Period	A	B
0	-300	-480
1	-80	-45
2	-80 [+90]	-45 [+250]
3	-80 (+50*)	-45
4		-45
5		-45
6		-45 (+60*)

*(salvage value) at end of life; [market value] at end of 2 years; Values in '000

$PW(15\%)_A = -300 - 80\,(P/A, 15\%, 2) + 90\,(P/F, 15\%, 2)$
$= -362$

$PW(15\%)_B = -480 - 45\,(P/A, 15\%, 2) + 250\,(P/F, 15\%, 2)$
$= -364$

Choose A which has a least negative PW

The market value for A at t=2 is higher because it is assumed that it is sold earlier (compared to assumed salvage value at t=3). The same logic applies for B but its overall PW is more negative

Non-Mutually Exclusive Projects

So far, we have looked at **mutually exclusive projects.** As detailed previously, projects can also be:

- Independent
- Contingent

When some projects are non-mutually exclusive, its necessary to enumerate alternatives to find mutually exclusive alternatives and make comparisons. A mechanism is needed to budget and allocate capital properly among alternatives.

Often, a company has too many non-mutually exclusive project proposals, and many constraints on available capital to allocate. Explicit enumeration can be demanding and even intractable.

Optimisation can help obtain maximum return based on limited capital and other constraints. **Linear programming** can be used to optimise a linear objective function, subject to one or more linear constraint equations. It can be used for project evaluation and selection.

The **capital allocation** problem can be represented mathematically as a linear objective function:

$$\max\ Z = Net\ PW = \sum_{j=1}^{m} B_j^* X_j$$

Where:

- Z = objective function
- B_j^* = PW of investment opportunity j during planning period
- X_j = Decision variable for opportunity j (1: project undertaken, 0: project not undertaken)
- m = Number of mutually exclusive combination of alternatives under consideration
- c_{kj} = Cash outlay required for opportunity j in time period k
- C_k = Maximum cash outlay permissible in time period k

Typically, there are two constraints in capital budgeting problems:

1. Limitations on cash outlays for period k of the planning horizon (i.e. budget):

$$\sum_{j=1}^{m} c_{kj} X_j \le C_k$$

2. Interrelationships among the projects. For example (note each project is worth 1):
    a. If projects p, q, r, are mutually exclusive then:

$$X_p + X_q + X_r \leq 1$$

    b. If project r can be undertaken only if project s has already been selected (contingent), then:

$$X_r \leq X_s \; or \; X_r - X_s \leq 0$$

    c. If project u and v are mutually exclusive and project r is dependent (contingent) on the acceptance of u or v then:

$$X_u + X_v \leq 1 \quad and \quad X_r \leq X_u + X_v$$

This can be solved using modelling/optimisation software like AIMMS or Excel solver.

If we look at an example: Considering a capital allocation problem with cash flow and PW value as shown in the following table. MARR=15% and total investment fund avaliable is £20,000. What is the right combination of projects to maximise present worth?

Project	Initial investment	Annual net cash flow	Project life, years
A	-8,000	3,870	6
B	-15,000	2,930	9
C	-8,000	2,680	5
D	-8,000	2,540	4

1. Calculate the NPW for the 4 projects:

$$PW_A = -8{,}000 + 3{,}870\,(P/A, 15\%, 6) \quad = \$6{,}646$$
$$PW_B = -15{,}000 + 2{,}930\,(P/A, 15\%, 9) \quad = -\$1{,}019$$
$$PW_C = -8{,}000 + 2{,}680\,(P/A, 15\%, 5) \quad = \$984$$
$$PW_D = -8{,}000 + 2{,}540\,(P/A, 15\%, 4) \quad = -\$748$$

2. Formula the equations:

$$Z = 6{,}646X_A - 1{,}019X_B + 984X_C - 748X_D$$

3. Subject to constraints:

$$8{,}000X_A + 15{,}000X_B + 8{,}000X_C + 8000X_D \leq 20{,}000$$
$$X_A,\ X_B,\ X_C \text{ and } X_D = 1 \text{ or } 0$$

4. Then use the 'solver' under data in Excel.

Cost of Capital

Now onto the final area of this section, the cost of capital. This refers to the rate of return expected by those parties contributing to the capital for a given level of risk.

If it is a loan, the cost of capital is the interest rate, and if it is equity, the cost of capital is the expected or promised dividend. We have already seen some of this in a previous section when calculating the weighted average cost of capital (WACC).

For (short term) **loans,** since interest paid for loans are tax-deductible, the effective after-tax cost of capital for a loan C_L (in %).

$$C_L = \left[\left(1 + \frac{r}{m}\right)^m - 1\right](1 - t)$$

Where $r = i_L$=nominal annual interest rate, m=number of compounding periods per year, and t=effective annual income tax rate. And if $m=1$ (i.e. no compounding within a year), then:

$$C_L = i_L(1 - t)$$

For **bonds**, these are usually issued on face value, which is the amount that investors will receive at the end of a specified period. In addition to this, the bondholder receives interest.

So say you buy a bond at £50, you would expect to receive the £50 back when the bond matures (at the end of a specified period, say 5 years), and during that 5 years, you will receive interest/yield, say at 2%.

Therefore, the cost of capital for bonds comes from:

- Interest/yield a company must pay to bondholders,
- A cut that an investment bank must take when issuing the bonds for the company
- If the maturity price of the bond given to the bondholder is higher than the selling price to the bondholder, the difference is a cost

For **equity,** companies' issue this in the form of stocks to their owners. For publicly listed companies, anyone can buy listed stocks and become a shareholder. Shareholders will not suffer losses greater than the amount invested when buying the stocks (i.e. limited liability). There are many types of stocks, but two common types are:

- Common: ordinary ownership, no guarantee of privileged over profits
- Preferred: stocks with privileges over common stocks in terms of sharing profits, often with guaranteed dividends (similar to bonds)

The owners of a common stocks will receive cash dividend declared by the company, as well as capital gain if/when sold. It is hard to determine the cost of capital for stocks, due to uncertainty in dividend, future stock price, and other macro and micro factors.

There are no tax savings for equity capital because dividends paid to shareholders are not tax-deductible. There are two common models to determine the capital cost of equity:

- Dividend Valuation Model
- Capital Asset Pricing Model (CAPM)

Let's look at the **Dividend Valuation Model** First. If we let Div_k = after-tax value of cash dividends received during year *k*. The current value of common stock is approximately *PW* of future cash receipts during an N-year ownership period:

$$P_o \approx \frac{Div_1}{(1+e_a)} + \frac{Div_2}{(1+e_a)^2} + \frac{Div_3}{(1+e_a)^3} + \ldots + \frac{Div_N}{(1+e_a)^N} + \frac{P_N}{(1+e_a)^N}$$

Where:

- P_o = current value of a share of common stock
- P_N = selling price of a share of common stock at the end of N years
- e_a = rate of return per year required by common stockholders, or after-tax cost of equity to the corporation. This must be sufficient to compensate shareholders for time value of money and risk believed to be associated with the investment.

The model assumes that dividends are constant over the life of the corporation. Hence, the current price of a share equals the *PW* of an infinite series of constant dividends plus the final selling price.

We can observe from the formula that the denominator will increase to infinity, and so the last term will be 0. And that the formula can be rewritten using the familiar notation if there is a constant *Div (D).* This equation is similar to that of a uniform payment cash flow series in the 'time value' chapter, which simplifies to a factor by which to multiply with. We can make e_a the subject of this:

$$\frac{P_N}{\infty} = 0$$

$$\underbrace{P_o = \frac{Div_1}{(1+e_a)} + \frac{Div_2}{(1+e_a)^2} + \frac{Div_3}{(1+e_3)^3} + \ldots + \frac{Div_N}{(1+e_a)^N} + \frac{P_N}{(1+e_a)^N}}$$

$$P_o = D[P/A, e_a, \infty] \qquad P_o = \frac{Div}{e_a} \qquad e_a = \frac{Div}{P_o}$$

When the price of the stock is assumed to *grow* at rate *g %*, then cost of capital becomes:

$$e_a = \underbrace{\frac{Div}{P_o}}_{\text{Dividend yield}} + \underbrace{g}_{\text{Capital yield}}$$

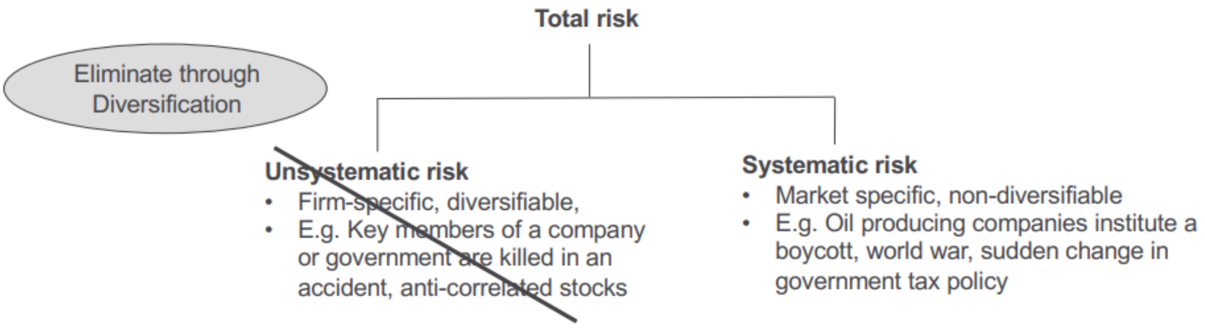
Let's look at an example where we apply the dividend valuation model. Say that a company plans to raise capital for a new plant by selling £2,500,00 worth of common shares, worth £20 each. If a 5% dividend is planned for the first year, and an appreciation of 4% in stock value is needed to attract investors, what is the actual cost of capital?

$$e_a = dividend\ yield + capital\ yield$$

Cost of capital is therefore 5% + 4% = 9%

The second type of model to determine the cost of equity is the **Capital Asset Pricing Model** (CAPM). This assumes that:

- Diversification reduces 'unsystematic' risk
- Investors expect return commensurate with level of risk
- Since investors can minimise unsystematic risk, the only risk determining risk premium is systematic risk
- Hence, there is no reward associates with unsystematic risk because it can be eliminated through diversification



β (beta) of a companies stock is usually published periodically by stockbrokers. This is a measure of risk/volatility:

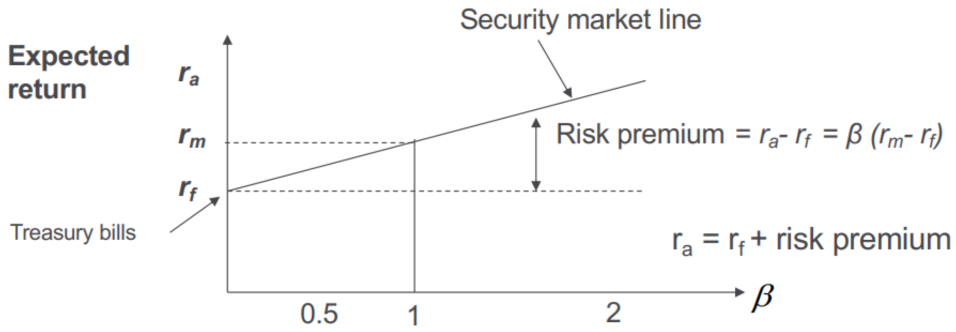
- $\beta < 1$: stock is less risky or less volatile in relation to market movement, so the resulting premium is smaller
- $\beta > 1$: stock is riskier, or more volatile in relation to market movement, which means a higher premium is required
- $\beta = 1$: stock has same risk (or same volatility, i.e. rise and fall the same percentage as the market, an average systematic risk) as market portfolio

The formula for an equity's expected return, r_e, is given by:

$$r_e = r_f + \beta(r_m - r_f)$$

Where r_f is the expected return on a risk free investment (e.g government bond), r_m is the return on stock in a defined market portfolio e.g. FTSE100 index.

CAPM says that in a competitive market, the expected (systematic) risk premium varies in direct proportion to β. Optimal investments in a market portfolio and risk-free asset can be plotted along the sloping line, know as the *Security market line*.



So to summarise, calculating the cost of capital can be done through either of these methods. Using the dividend valuation model or the CAPM model allows us to find a (%) cost of capital that could then be used as the MARR when analysing investment alternatives.